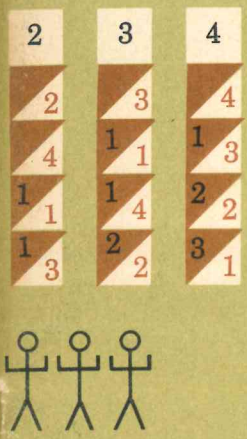
Understanding Numeration Systems



$10_{twelve} = 12$

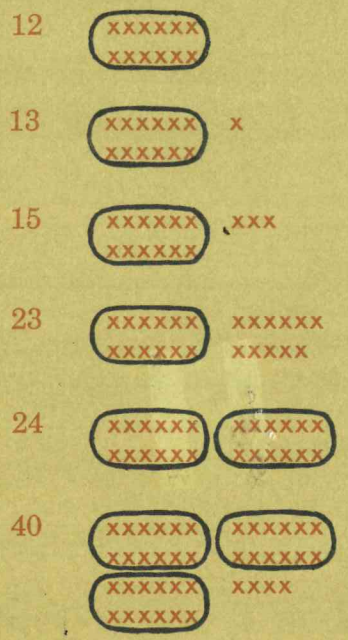
$11_{twelve} = 13$

$13_{twelve} = 15$

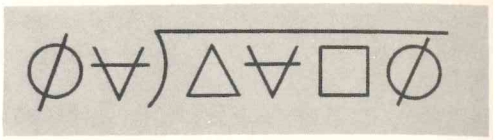
$1E_{twelve} = 23$

$20_{twelve} = 24$

$34_{twelve} = 40$



EXPLORING MATHEMATICS ON YOUR OWN



"There was an old man who said, 'Do
Tell me how I should add two and two.
I think more and more
That it makes about four —
But I fear that is almost too few.'"
—Anonymous

BEFORE YOU BEGIN TO READ

Reading about mathematics can be as exciting an adventure as reading a mystery story or exploring a cave. There are many surprises, puzzles, tricks, and interesting ideas in mathematics. If you will do some exploring in mathematics by yourself, you will enjoy discovering new ideas. One of the most interesting topics in mathematics is the study of numeration systems and their bases.

You will probably need to read this booklet on numeration systems differently from the way you would read a story, for example. To begin with, you should read it slowly. Don't be surprised if you don't quite understand every sentence or every paragraph the first time. But have patience. Get into the habit of reading mathematics with a pencil and paper within easy reach. Don't hesitate to use them. If you work the exercises, make the drawings, and complete the problems, it will be easier for you to understand what you read.

We hope that this booklet will enable you to share the pleasure others have had in exploring numeration systems.

Andrea Pastirko

EXPLORING MATHEMATICS ON YOUR OWN

Understanding Numeration Systems

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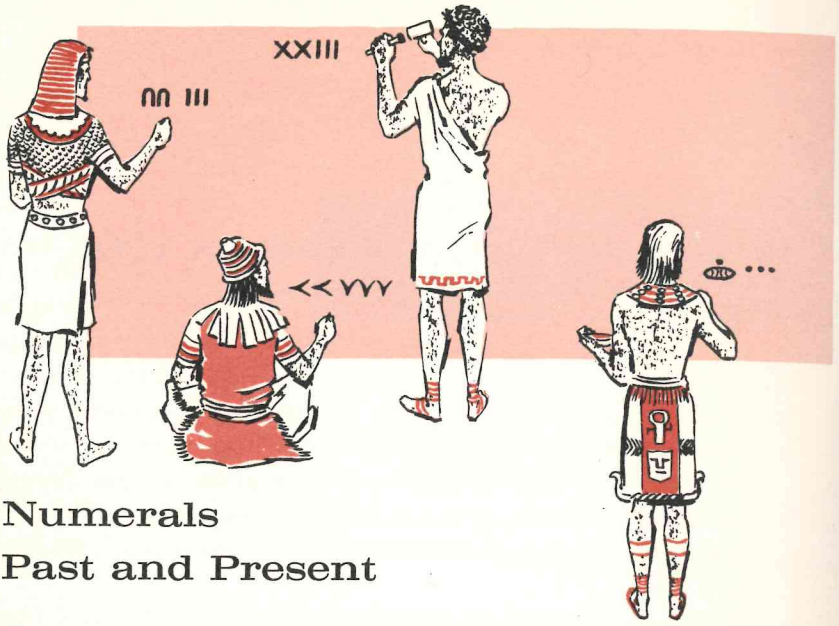


Numerals Past and Present

Numerals, Numbers, and Systems

Probably the greatest invention ever made by man was a number system. Although it is simple for us to use the symbol 23 to represent twenty-three objects, we should realize that it took man many centuries to build the system in which the figures 2 and 3, arranged in that order, do stand for twenty-three things.

The numbering system we use today was not the only one or the first one developed by man. Here are some of the ways that 23 has been written in the past:

Egyptian

Babylonian

Roman XXIII

Mayan

The amount represented by the symbols above is the same, but the symbols used are different. The symbols used to represent numbers are called *numerals*. A numeral, then, is a name or symbol used to stand for a number. When a certain set of symbols is used

to represent numbers, a *numeration system* is formed. In a numeration system it is possible to use only a few symbols to represent many different numbers. This is done by arranging different combinations of the symbols and by giving importance to the order in which the symbols are written.

The numeration system we use is called the *Hindu-Arabic system*. It has this name because it was invented in India and brought to Europe by the Arabs. Most people believe that our system of numerals is a "natural" system, or the best system, or the only system we need. This is not true. In fact, in our modern world of atomic energy, space travel, and electronic computing machines we often find a need for different numeration systems. Some day someone may invent a numeration system far better than our present one — just as our system was far better than the Roman system.

If we consider other numeration systems, like the ones that we will describe in the following pages, we will have a better understanding of our numerals, how our computation processes work, and how new systems are being used in science.

Understanding Our Numeration System

Let us now take a close look at the numbers we use every day. Our numeration system has ten basic symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These symbols are called *digits*. We use combinations of our ten number symbols to write numerals for very large or very small numbers. This is possible because we assign different values to a digit, depending upon its position in a numeral, or its *place value*. We can illustrate this with a counting example. When we count in our system, we usually group by tens. In Figure 1, we have one group of ten figures, and three left over. Thus, these figures are counted to be 13. The numeral 13 means $(1 \times 10) + (3 \times 1)$. When we count, we usually work with "whole things" and thus we say that 1, 2, 3 . . . represent whole numbers.

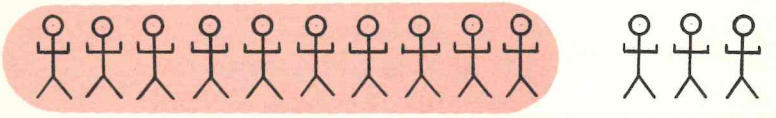


Figure 1

Consider now the large group of x's in Figure 2. We can draw lines around groups of ten until fewer than ten remain. Then we can draw a line around a group of ten tens or a hundred.

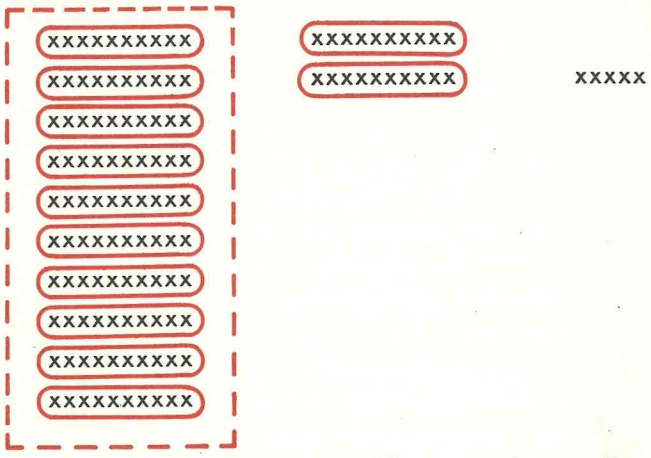


Figure 2

The number of x's can be represented by the numeral 125, which means $(1 \times 10 \times 10) + (2 \times 10) + (5 \times 1)$.

When we do addition problems, we often have to "regroup" numbers to form groups of ten. We often speak of this regrouping as "carrying" to the next digit place.

$$\begin{aligned} 18 + 5 &= (10 + 8) + (2 + 3) \\ &= 10 + (8 + 2) + 3 \\ &= 10 + 10 + 3 \\ &= 23 \end{aligned}$$

We have seen that our numeral system uses ten basic symbols and that the size of the group represented by the symbol depends upon its position or place value. Thus, 4 is the numeral for a set of four things, but each 4 in the numeral 444 has a different value. Each place has a value 10 times as much as the place on its right:

$$444 = (4 \times 10 \times 10) + (4 \times 10) + (4 \times 1).$$

This could be written as:

$$444 = (4 \times 10^2) + (4 \times 10^1) + (4 \times 1).$$

The 2 and the 1 written just above the 10's are called *exponents*, and show how many 10's are multiplied together.

In the same way, a decimal that expresses a part of a whole thing has place value:

$$.333 = \left(3 \times \frac{1}{10}\right) + \left(3 \times \frac{1}{10} \times \frac{1}{10}\right) + \left(3 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}\right)$$

Since our system of writing numerals uses ten symbols and the principle of grouping by ten, we say that it is a *base ten* system. It is also called a *decimal* system, the word "decimal" coming from the Latin word for ten.

Our decimal system has two important features. It uses the place value principle that we have just discussed, and it has a symbol for zero. The symbol 0 is used to record "no" things. Zero also serves as a place holder, just like any other digit. Without a zero, it would be impossible to distinguish between numerals like 4 and 40. The features of our numeral system enable us to represent numbers concisely and to perform calculations easily. Simple multiplication and division problems now done by a nine-year-old child in a few minutes took the ablest Roman mathematicians of two thousand years ago several hours to do.

EXERCISE SET 1

Base Ten Numerals

1. What is the value of the 3 in each of these numerals?

a. 403,157 3,000 b. .0143 3/10,000 c. 2.372 3/10

2. What is the largest whole number you can write using the digits 2, 3, 4? 432

3. What is the smallest whole number you can write using the digits 4, 3, 2? 234

4. Make tally marks and group them to show a picture of place value for 137.

5. Show the value of each digit in these numbers by showing the multiple of ten each represents. For example:

$$478 = (4 \times 10 \times 10) + (7 \times 10) + (8 \times 1)$$

a. 5,684 b. 2,070 c. .413

6. a. Can you think of a reason why ten is a base for many numeral systems? 10 fingers

b. The Maya Indians of Central America had a numeral system with a base of twenty. Can you think of a reason for their selection of twenty as a base? 10 fingers 10 toes

Numeration with Five Symbols

Counting with Fives

Picture a primitive man, spear in hand, stalking a herd of wild animals. When he overtakes the herd, he makes a count of animals, for he wants to report on this herd to his village. Since he has a spear in one hand, he counts on the fingers of the other hand only. Perhaps it was in this way that some primitive peoples developed base five numeral systems.

Let's try to build a base five numeral system. We will need five number symbols, so let's use the numerals 0, 1, 2, 3, 4. When we count in this system, we will group by fives.

Figure 3

The number of *x*'s in Figure 3 are separated into two groups of five with 3 left over. In the base five system, we would write this 23_{five}. We write the "five" to show that this number is not "twenty-three" as we usually think of it in base ten. It is 2 fives and 3 ones. The numeral in the second place to the left tells us how many groups of five are tallied. In our base ten or decimal numerals, the number of objects would be written as 13, which means one group of ten and three more. In a similar way, 41_{five} means $(4 \times \text{five}) + 1$, or 4 groups of five and one more. And, 324_{five} means $(3 \times \text{five} \times \text{five}) + (2 \times \text{five}) + 4$. The value of each digit place in a base five numeral is five times as much as the place on its right. We can invent numerals which will have any place value we may want them to have.

Whenever we write a numeral like 24, we will consider it a base ten numeral. Numerals in other bases will have the base indicated to the right and below the numeral, as shown in the following examples. These indicators are called *subscripts*.

$$24 = (2 \times 10) + 4$$
$$24_{five} = (2 \times \text{five}) + 4$$

Let's make a closer comparison of base ten and base five numerals.

Base ten notation	Base five grouping	Base five notation	Base five names
0		0	zero
1	x	1	one
2	xx	2	two
3	xxx	3	three
4	xxxx	4	four
5	(xxxxx)	10_{five}	one five
6	(xxxxx) x	11_{five}	one five and one
7	(xxxxx) xx	12_{five}	one five and two
8	(xxxxx) xxx	13_{five}	one five and three
9	(xxxxx) xxxx	14_{five}	one five and four
10	(xxxxx) (xxxxx)	20_{five}	two-fives
16	(xxxxx) x (xxxxx) (xxxxx)	31_{five}	three-fives and one
24	(xxxxx) xxxx (xxxxx) (xxxxx) (xxxxx)	44_{five}	four-fives and four
25	[(xxxxx) (xxxxx) (xxxxx) (xxxxx) (xxxxx)]	100_{five}	five-squared
31	[(xxxxx) (xxxxx) (xxxxx) (xxxxx) (xxxxx)] (xxxxx) x	111_{five}	five-squared, one five and one
50	[(xxxxx) (xxxxx) (xxxxx) (xxxxx) (xxxxx)] [(xxxxx) (xxxxx) (xxxxx) (xxxxx) (xxxxx)]	200_{five}	two-five-squared

When a base five numeral is written according to the place value of each symbol, it can easily be changed to base ten notation.

$$\begin{aligned} 342_{\text{five}} &= (3 \times \text{five} \times \text{five}) + (4 \times \text{five}) + 2 \\ &= (3 \times 25) + (4 \times 5) + 2 \quad \text{[base ten numerals]} \\ &= 75 + 20 + 2 = 97 \end{aligned}$$

A good way to keep place value straight when changing from base five to base ten is to refer to quarters, nickels, and pennies. For example, 342_{five} can be interpreted as:

$(3 \times \text{five} \times \text{five}) + (4 \times \text{five}) + 2,$
or 3 quarters + 4 nickels + 2 pennies,
or 75 cents + 20 cents + 2 cents = 97 cents.

Another way to remember the place value of base five numerals is to write the symbols in columns. Let's use this method to get base ten interpretations for 34_{five}, 241_{five}, 303_{five}, $1{,}420_{\text{five}}$, and $3{,}042_{\text{five}}$.

five x five x five (five^3)	five x five (five^2) or "quarters"	five x one or "nickels"	ones or "pennies"	
		3	4	$34_{\text{five}} = (3 \times \text{five}) + 4$ $= 19$
	2	4	1	$241_{\text{five}} = (2 \times \text{five} \times \text{five}) +$ $(4 \times \text{five}) + 1$ $= 71$
	3	0	3	$303_{\text{five}} = (3 \times \text{five} \times \text{five}) +$ $(0 \times \text{five}) + 3$ $= 78$
1	4	2	0	$1{,}420_{\text{five}} = (1 \times \text{five} \times \text{five} \times$ $\text{five}) + (4 \times \text{five} \times$ $\text{five}) + (2 \times \text{five}) + 0$ $= 235$
3	0	4	2	$3{,}042_{\text{five}} = (3 \times \text{five} \times \text{five} \times$ $\text{five}) + (0 \times \text{five} \times$ $\text{five}) + (4 \times \text{five}) + 2$ $= 397$

EXERCISE SET 2

Understanding Base Five Numerals

1. Draw x's and encircle groups to show the meaning of each of the following numerals in base ten:

a. 14 b. 23 c. 137

2. Draw x's and encircle groups to show the meaning of each numeral:

a. 14_{five} b. 23_{five} c. 134_{five}

3. Write the meaning of each of these base ten numerals according to place value; for example, $44 = (4 \times 10) + 4$:

a. 53 b. 123 c. 2,304

4. Write the meaning of each of these base five numerals according to place value; for example, $23_{\text{five}} = (2 \times \text{five}) + 3$:

a. 43_{five} b. 123_{five} c. $2{,}304_{\text{five}}$

5. Write the meaning of each of these base five numerals and change to base ten notation:

a. 24_{five} b. 441_{five} c. $1{,}203_{\text{five}}$

Changing from Base Ten to Base Five

We have already changed numbers written in base five to base ten numerals. Let's now see if we can change from base ten to base five. Let's try to write 89 as a base five numeral.

In base five, the values of the places are one, five, five $\times$ five, five $\times$ five $\times$ five, etc. In base ten, these place values are:

one = 1
five = 5
five $\times$ five = 25
five $\times$ five $\times$ five = 125

Now, 89 is less than 125; hence it is less than a four-place base five number. But it is more than 25, so it can be written as a three-place base five numeral.

$$\begin{array}{r} 3 \\ 25\overline{)89} \\ \underline{75} \\ 14 \end{array}$$

The division shows that there are 3 five $\times$ five groups in 89, and 14 units left over.

$$\begin{array}{r} 2 \\ 5\overline{)14} \\ \underline{10} \\ 4 \end{array}$$

This division shows that there are 2 five groups in 14, and 4 units left over.

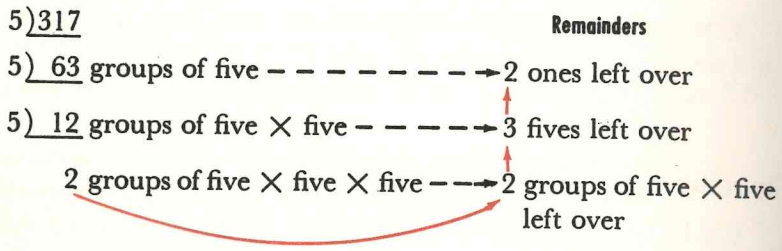
We can see that:

$$89 = (3 \times \text{five} \times \text{five}) + (2 \times \text{five}) + 4$$
$$= 324_{\text{five}}$$

We can think of this process as one in which the base ten numeral is broken down into multiples of five. Thus:

$$317 = 250 + 50 + 15 + 2$$
$$= (2 \times \text{five} \times \text{five} \times \text{five}) + (2 \times \text{five} \times \text{five}) + (3 \times \text{five}) + 2$$
$$= 2{,}232_{\text{five}}$$

A shortcut for changing base ten to base five is to make continued divisions. (In the following example, division is indicated with this sign:)____.)


5)317
Remainders
5) 63 groups of five
2 ones left over
5) 12 groups of five × five
3 fives left over
2 groups of five × five × five
2 groups of five × five left over

Writing the remainders in reverse order gives the result $317 = 2{,}232_{\text{five}}$.

EXERCISE SET 3

Changing to and from Base Five

1. Change these base ten numerals to base five:
   a. 38 b. 64 c. 156 d. 377
2. Change these base five numerals to base ten:
   a. 23_{five} b. 222_{five} c. 340_{five} d. $2{,}003_{\text{five}}$

A Base Five Counting Board

In many Oriental shops today, no cash registers or adding machines are seen. Instead, the merchant totals his sales on an instrument consisting of beads that can be made to slide on slender bamboo rods. Such an instrument is pictured in Figure 4. It is

called a counting board or *abacus*. One type of abacus was used in China as early as the sixth century B.C., and a form of this counting board has been used at some time in almost every area of the world.

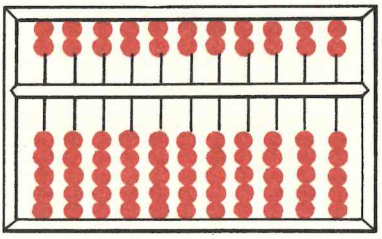


Figure 4

You can make a base five abacus that will illustrate place value in base five and help in computations with base five numerals. You will need these materials: one $1'' \times 1'' \times 8''$ wood stick, 5 long nails, twenty-five $1'' \times 1''$ cardboard squares or beads (5 of each color if possible).

To construct this abacus, first locate five equally-spaced points about $1\frac{1}{2}''$ apart along the middle of one side of the stick, as shown in Figure 5*a*. Punch holes in the center of each cardboard square. Place five cardboard squares of the same color on each nail and pound the nails into the sticks at the marked points.

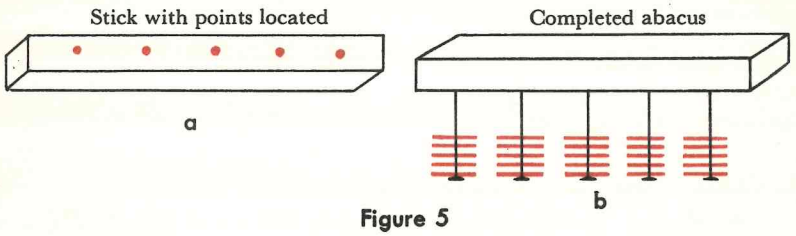


Figure 5

This counting board can be used to represent base five numerals. Figure 6 shows how we can use the abacus to represent 12_{five}.

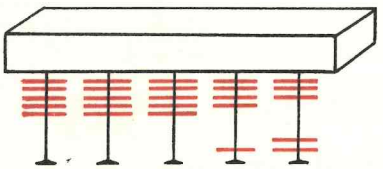
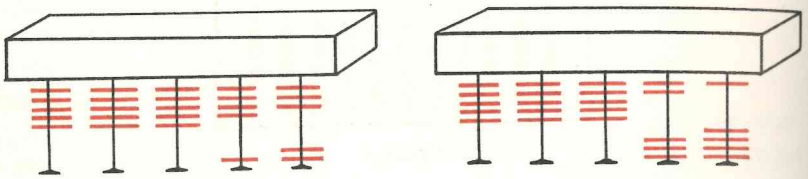


Figure 6

We can now use the abacus for base five computations.

Addition of Numbers Written in Base Five

You know that the sum of 12 and 22 is 34, but what is the sum of 12_{five} and 22_{five}? Since addition is essentially a form of counting, we can use the base five counting board to do this sum. To add 12_{five} and 22_{five}, move the squares as shown in Figure 7.

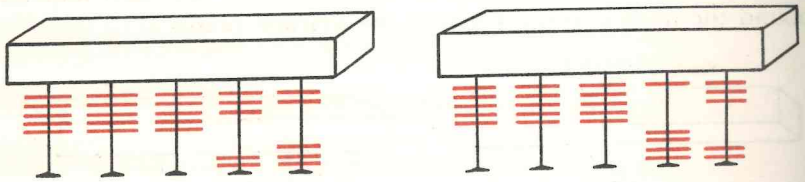


12_{five} $\qquad$ $12_{five} + 22_{five} = 34_{five}$

Figure 7

We see that the result is comparable to that obtained from a base ten computation of 12 and 22.

Now try $23_{five} + 14_{five}$. The abacus drawings in Figure 8 show that when you add 3_{five} and 4_{five}, you "carry" a five to the second digit place.



23_{five} $\qquad$ $23_{five} + 14_{five} = 42_{five}$

Figure 8

Another way to add small numbers in base five numeration is to count on a number line. The base five number line in Figure 9 shows how to add $4_{five} + 3_{five}$.

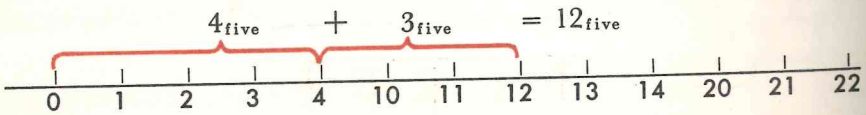


Figure 9

If you want to become proficient in adding numbers written in base five, you should make an addition table for reference or else memorize all the addition combinations, just as you did in school

for base ten. A good way to check your work is to translate to base ten and compare your results as shown in the following examples. Notice the similarity of the base five and base ten addition processes.

A.

		Check
23_{five} =	2 fives and 3 ones =	13 = 1 ten and 3 ones
$+\ 13_{five}$ =	+ 1 five and 3 ones =	+ 8 = + 8 ones
41_{five} =	3 fives and 6 ones =	21 = 1 ten and 11 ones
	or	or
	4 fives and 1 one	2 tens and 1 one

B.

	Check
24_{five} =	14
$+\ 13_{five}$ =	+ 8
42_{five} =	22

C.

	Check
32_{five} =	17
14_{five} =	9
$+\ 43_{five}$ =	+ 23
144_{five} =	49

EXERCISE SET 4

Base Five Addition

1. Copy and complete this addition table for numbers written in base five:

+	1	2	3	4	10
1	2	3	4	10	11
2	3	4	10	11	12
3	4	10	11	12	13
4	10	11	12	13	14
10	11	12	13	14	20

2. Use this table to perform these additions in base five.

a. $23_{five} + 21_{five}$ = 44

b. $22_{five} + 14_{five}$ = 41

c. $43_{five} + 14_{five}$ = 112

d. $244_{five} + 204_{five}$ = 1003

3. How much do you "carry" when you add these numbers in base ten?

59
27

a ten

4. How much do you "carry" when you add these numbers in base five?

23_{five}
14_{five}

a five

Subtraction in Base Five

$12_{five} - 4_{five} = ?_{five}$

What is subtraction? You probably know that subtraction is the inverse of addition. In other words, in base ten if $3 + 4 = 7$, then $7 - 4 = 3$ and $7 - 3 = 4$, or if $a + b = c$, then $c - b = a$ and $c - a = b$. In the subtraction $7 - 4$, we are asking what number added to 4 gives 7.

If we know the addition combinations for numbers written in base five, we can easily subtract numbers in base five. In the subtraction problem $12_{five} - 4_{five}$, we are asking what number added to 4_{five} will give 12_{five}. Looking in the addition table of Exercise Set 4, we find 12_{five} in row 3. This tells us that $4_{five} + 3_{five} = 12_{five}$, and we therefore know that $12_{five} - 4_{five} = 3_{five}$.

Study the following subtraction examples, and check the results by translating to base ten. Notice the similarity between the base five and base ten subtraction methods.

A.

40_{five} = 4 fives and 0 ones = 20 = 2 tens and 0 ones = 1 ten and 10 ones

$-\ 3_{five}$ = $-$ 3 ones = $-\ 3$ = $-$ 3 ones = $-$ 3 ones

32_{five} = 17 = 1 ten and 7 ones

B.	C.	D.	E.	Check
14_{five}	11_{five}	142_{five}	403_{five} =	103
$-\ 3_{five}$	$-\ 4_{five}$	$-\ 24_{five}$	$-\ 134_{five}$ =	$-\ 44$
11_{five}	2_{five}	113_{five}	214_{five} =	59

Another way to do base five subtraction is to label a number line with base five numerals and count backward.

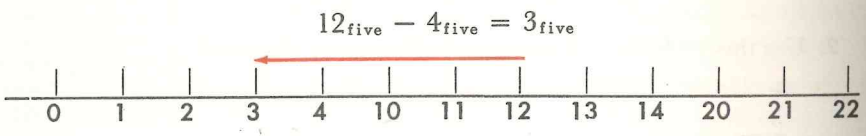


Figure 10

Then $12_{five} - 4_{five}$ becomes a matter of beginning at 12_{five} and counting 4 units to the left. This brings us to the answer, 3_{five}. Or, if we start at 4_{five} and count the units to 12_{five}, we find that the difference between the two is 3_{five}.

EXERCISE SET 5

Base Five Subtraction

1. Perform the following subtractions. Check by adding with the base five abacus or by translating to base ten:

a.	b.	c.	d.
44_{five}	32_{five}	320_{five}	$4,103_{\text{five}}$
$-\ 23_{\text{five}}$	$-\ 14_{\text{five}}$	$-\ 42_{\text{five}}$	$-\ 424_{\text{five}}$

2. When we "borrow" in the problem $32 - 14$, with base ten numerals, how many units do we "borrow?"

3. When we "borrow" in base five, in the problem $32_{\text{five}} - 14_{\text{five}}$, how many units do we "borrow?"

Base Five Addition and Subtraction with a Nomograph

If you have trouble doing additions or subtractions involving base five numerals, the chart in Figure 11, called a *nomograph*, can help you.

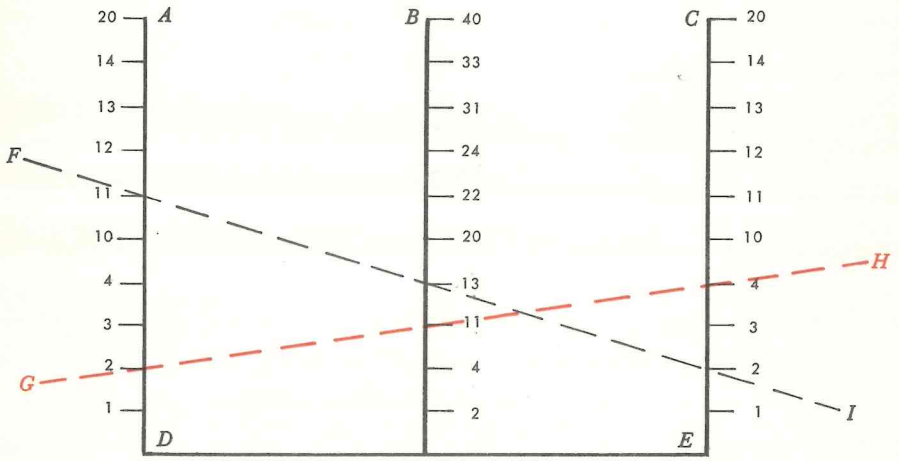


Figure 11

A nomograph is very easy to make. First draw a horizontal line *DE* on a sheet of graph paper. Then draw three parallel, vertical lines the same distance apart to intersect line *DE*. Label the lines *A*, *B*, and *C*, as illustrated. Mark off equal spaces on lines *A*, *B*, and *C*. Starting with *D* and *E* as zero points, number the corresponding marks on lines *A* and *C* consecutively with base five

numerals. Number center line B in base five also, but give each mark on this line a numerical value twice that of the corresponding marks on scales A and C.

With a straight edge, connect any point on A with any point on C. The sum of the two numbers represented by the points will be found at the point where the straight edge crosses the center line B. The A reading plus the C reading is equal to the B reading, or $A + C = B$. For example, line FI shows that $11_{five} + 2_{five} = 13_{five}$. The nomograph can also be used for subtraction, for $A = B - C$ and $C = B - A$. Line GH shows that $11_{five} - 4_{five} = 2_{five}$ and $11_{five} - 2_{five} = 4_{five}$. If you have studied plane geometry, you can easily show why the nomograph "works."

EXERCISE SET 6

Working with the Base Five Nomograph

1. Use your nomograph to find the following sums:
   a. $12_{five} + 3_{five}$ 20 b. $11_{five} + 14_{five}$ 24 c. $13_{five} + 13_{five}$ 31
2. Use your nomograph to find the following differences:
   a. $13_{five} - 10_{five}$ 3 b. $11_{five} - 3_{five}$ 3 c. ~~23~~ 24 $_{five} - 14_{five}$ 4

Multiplication in Base Five

If you were given the problem, "Find the sum of $3 + 3 + 3 + 3$ in base ten," you would probably take a short cut and use the multiplication fact $4 \times 3 = 12$. We can make short cuts for repeated additions in base five by constructing a multiplication table with base five numerals. For example, $3_{five} + 3_{five} + 3_{five} + 3_{five} = 4_{five} \times 3_{five} = 22_{five}$. Copy and complete the following base five multiplication table.

Multiplication Table — Base Five

×	1	2	3	4	10
1	1	2	3	4	10
2	2	4	11	13	20
3	3	11	14	22	30
4	4	13	22	31	40
10	10	20	30	40	100

Napier's Bones for Base Five Multiplication

In 1617, a Scottish mathematician named John Napier introduced a device, now called Napier's bones, for performing multiplications. We can make a set of Napier's bones to do multiplications in base five.

Cut six strips of cardboard (or plywood), each one inch by five inches. Mark five strips into squares and triangles as shown in Figure 12*a*, and label these five strips with the base five products for each digit, as shown in Figure 12*b*. Make the remaining strip an index, as shown in Figure 12*c*.

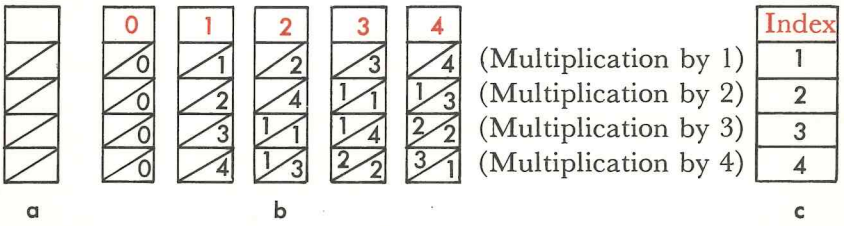
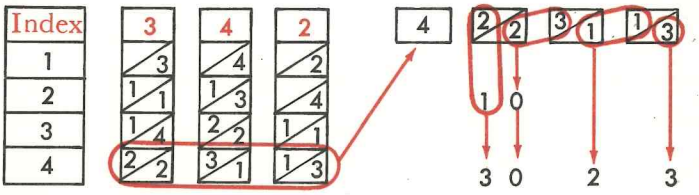


Figure 12

We can use these strips to solve multiplication problems, such as $342_{\text{five}} \times 4$. Select the strips or "bones" for 3, 4, and 2, and place them in order next to the index, as in Figure 13. Since we are multiplying by 4, we obtain the product from the 4 row. Bring down the 3 on the extreme right, and then add the remaining figures diagonally in order to get the rest of the digits in the product.



$$\begin{array}{r} 342_{\text{five}} \\ \times\ 4 \\ \hline 3023_{\text{five}} \end{array}$$

Figure 13

Notice the similarity between base ten and base five multiplication processes as you study the following problems. Check the work by using Napier's bones or by translating to base ten numerals.

A.

$$\begin{array}{r} 32_{\text{five}} \\ \times\ 2 \\ \hline 114_{\text{five}} \end{array}$$

B.

$$\begin{array}{r} 34_{\text{five}} \\ \times\ 3 \\ \hline 212_{\text{five}} \end{array}$$

C.

$$\begin{array}{r} 243_{\text{five}} \\ \times\ 42_{\text{five}} \\ \hline 1041 \\ 2132 \\ \hline 22411_{\text{five}} \end{array}$$

D.

$$\begin{array}{rcr} 41_{\text{five}} & = & 21 \\ \times\ 24_{\text{five}} & = & \times\ 14 \\ \hline 314 & & 84 \\ 1320 & & 210 \\ \hline 2134_{\text{five}} & = & 294 \end{array}$$

Check

EXERCISE SET 7

Base Five Multiplication and Numeration System Principles

1. Find these products:

a. $\begin{array}{r} 32_{\text{five}} \\ \times\ 3 \\ \hline \end{array}$ b. $\begin{array}{r} 43_{\text{five}} \\ \times\ 4 \\ \hline \end{array}$ c. $\begin{array}{r} 34_{\text{five}} \\ \times\ 23_{\text{five}} \\ \hline \end{array}$ d. $\begin{array}{r} 103_{\text{five}} \\ \times\ 42_{\text{five}} \\ \hline \end{array}$

2. In base ten, we know that the product of two numbers is the same, whether the first is multiplied by the second, or the second is multiplied by the first; for example, $6 \times 4 = 4 \times 6$. Multiplication in base ten is said to be *commutative*, for we can commute (interchange) the members of a product and still obtain the same numerical result.

a. Is base five multiplication commutative?

b. Is base ten addition commutative?

c. Is base five addition commutative?

3. Consider this multiplication: $3 \times 5 \times 6$. We could write the product as $(3 \times 5) \times 6 = 15 \times 6 = 90$, or $3 \times (5 \times 6) = 3 \times 30 = 90$. In both cases, we obtain the same result. This is an example of the *associative principle of multiplication* in base ten.

a. Is base five multiplication associative?

b. Is base ten addition associative?

c. Is base five addition associative?

4. What is the product of 8 and 17? You probably haven't memorized an answer for 8×17, so you might think this way: $8 \times 7 = 56$, carry 5, $8 \times 1 = 8$, $8 + 5 = 13$, and the answer is 136. This method of thinking could also be written in the following way:

$$8 \times 17 = 8(10 + 7) = 8 \times 10 + 8 \times 7 = 80 + 56 = 136$$

This illustrates an important principle of base ten numbers which is usually called the *distributive principle of multiplication over addition.* In

the above example, we distributed the multiplier 8 to the 10 and 7. Does this distributive principle apply to base five?

Division in Base Five

If we have a multiplication table, we can use it for division as well as for multiplication. This is based on the fact that division is the inverse of multiplication. Division "undoes" multiplication. If $3 \times 4 = 22_{\text{five}}$, then $22_{\text{five}} \div 4 = 3$, and $22_{\text{five}} \div 3 = 4$, or, in any base, if $a = b \times c$, then $a \div b = c$.

As you study the following examples, note that we do our base five divisions the same way as in the decimal system; that is, by comparing, multiplying, and subtracting. (All the numerals are base five, but the base indicator is not written in the body of the work.)

A.
$$\begin{array}{r} 103_{\text{five}}\ r\ 1 \\ 4\overline{)423_{\text{five}}} \\ 4 \\ \hline 23 \\ 22 \\ \hline 1 \end{array}$$

B.
$$\begin{array}{r} 114_{\text{five}}\ r\ 4 \\ 23_{\text{five}}\overline{)3241_{\text{five}}} \\ 23 \\ \hline 44 \\ 23 \\ \hline 211 \\ 202 \\ \hline 4 \end{array}$$

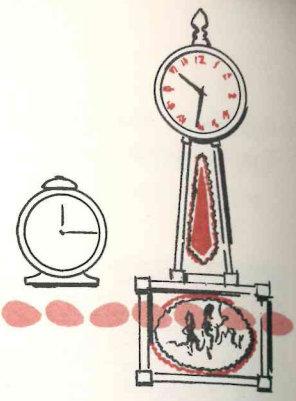
EXERCISE SET 8

Base Five Division

Perform the indicated divisions:

1. $3\overline{)341_{\text{five}}}$ 112
2. $4\overline{)310_{\text{five}}}$ 40
3. $21_{\text{five}}\overline{)441_{\text{five}}}$ 21
4. $32_{\text{five}}\overline{)3433_{\text{five}}}$ 104

Numeration with the Dozen System



Counting with Twelves

We have seen that it is possible to use a numeration system other than our base ten system. Although we have used five as a base, we could have used any other number. A very important question arises then: "Is there a numeration base better than base ten?" To answer this question, we must consider ways in which a different base could improve upon our present numeration method.

First, it would be practical to select a numeration base having more even divisors than 10 has, for this would simplify work with common fractions. A number like 12 might be a good base, for the even divisors, or factors, of 12 are 2, 3, 4, and 6. The only divisors of 10 are 2 and 5 (besides 10 itself and 1).

It would also be convenient to have a base that is related to our common units of measure. Many of our units of measure are based on 12 or multiples of 12; for example, 12 inches in 1 foot, 12 hours on the clock face, 60 minutes in an hour, 360 degrees in a circle, 12 eggs in a dozen, and 144 units in a gross.

About two hundred years ago, Georges Buffon, a French naturalist, suggested that a base twelve numeration system be universally adopted. Although the base twelve system is sometimes called the "dozen system," we usually call it the *duodecimal system*. *Duodecimal* is another word for twelve, just as *decimal* is another word for ten.

The fight for base twelve was carried into this century, and duodecimal societies sprang up all over the world. Today the Duodecimal Society of America recommends that we use a base twelve system instead of our present decimal notation.

In the duodecimal system we have twelve symbols and form groups of twelves (dozens), twelve-twelves (gross), and so on. This means we must have two new number symbols. The symbols usually used are T for ten and E for eleven.

In the duodecimal system, we group in dozens. This means that the marks in Figure 14 are counted 13_{twelve} because we have 1

Figure 14

group of a dozen and 3 ones. This numeral, 13_{twelve}, is read "one do three," which means one dozen and three.

Let's compare counting in the duodecimal system with counting in the decimal system.

Decimal notation	Duodecimal grouping	Duodecimal notation	Duodecimal names
0		0	zero
1	x	1	one
2	xx	2	two
3	xxx	3	three
4	xxxx	4	four
5	xxxxx	5	five
6	xxxxxx	6	six
7	xxxxxx x	7	seven
8	xxxxxx xx	8	eight
9	xxxxxx xxx	9	nine
10	xxxxxx xxxx	T	ten
11	xxxxxx xxxxx	E	eleven
12	xxxxxx xxxxxx	10_{twelve}	one do
13	xxxxxx xxxxxx x	11_{twelve}	one do one
23	xxxxxx xxxxxx xxxxxx xxxxx	$1E_{twelve}$	one do eleven
24	xxxxxx xxxxxx xxxxxx xxxxxx	20_{twelve}	two do
40	xxxxxx xxxxxx xxxxxx xxxxxx xxxxxx xxxx xxxxxx	34_{twelve}	three do four

We can use a place value table to show the relationship between base twelve and base ten numerals. The following table can help us get base ten interpretations for 45_{twelve}, 307_{twelve}, and $12E0_{twelve}$:

twelve x twelve x twelve ($twelve^3$) or dozen-dozen-dozen or great gross (1728)	twelve x twelve ($twelve^2$) or dozen-dozen or gross (144)	twelve x one or dozen or twelves	ones or units
		4	5
	3	0	7
1	2	E	0

$$45_{twelve} = (4 \times twelve) + (5 \times one) = 53_{ten}$$

$$307_{twelve} = (3 \times twelve \times twelve) + (0 \times twelve) + 7 \times 1 = 439_{ten}$$

$$12E0_{twelve} = (1 \times twelve \times twelve \times twelve) + (2 \times twelve \times twelve) + (E \times twelve) + (0 \times one)$$

$$= (1 \times 1728_{ten}) + (2 \times 144_{ten}) + (11_{ten} \times 12_{ten}) + (0 \times 1)$$

$$= 2148_{ten}$$

EXERCISE SET 9

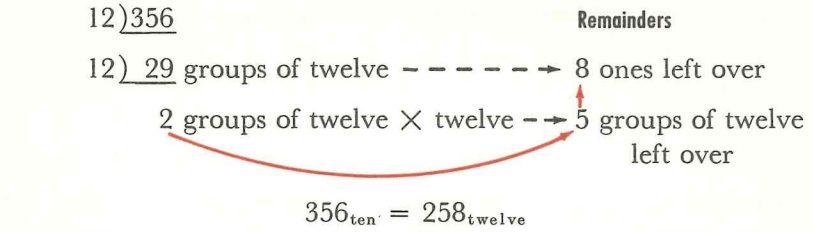
Working with Base Twelve

Change these numbers, written in base twelve, to base ten.

1. 45_{twelve} 53
2. 9_{twelve} 9
3. 37_{twelve} 43
4. 24_{twelve} 28

Changing from Base Ten to Base Twelve

If there are advantages in using base twelve numerals, it is important to know how to change numbers written in base ten to base twelve. This conversion is best done by repeated division, in the same way that we changed base ten numerals to base five. Let's change 356 to base twelve.



$356_{ten} = 258_{twelve}$

The remainders written in reverse order give the desired base twelve numeral.

EXERCISE SET 10

Changing to and from Base Twelve

1. Change these base ten numerals to base twelve:
   a. 78 66 b. 118 9T c. 445 311 d. 3,784 2234
2. Change these base twelve numerals to base ten:
   a. 78_{twelve} 92 b. TE_{twelve} 131 c. 275_{twelve} 377 d. $1T0_{twelve}$ 264

Addition and Subtraction in the Duodecimal System

After you have learned to count in a new numeration system, you are ready for addition problems. Don't forget that addition in base twelve is merely a short cut for counting with duodecimals. You might want to use a number line or a nomograph or a duodecimal abacus to help you solve addition problems. A duodecimal number line is shown in Figure 15.

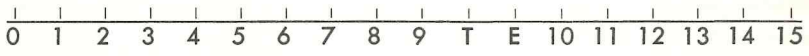


Figure 15

Figure 16 shows how the number line can be used to add T and 5_{twelve}.

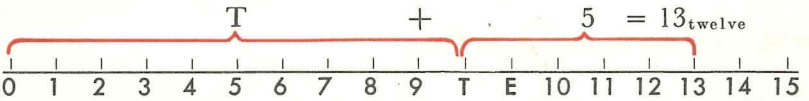


Figure 16

If you copy and complete the duodecimal addition table on the next page, you can use it to add numbers expressed in the duodecimal system without memorizing base twelve combinations.

Addition Table — Base Twelve

+	1	2	3	4	5	6	7	8	9	T	E
1	2	3	4	5	6	7	8	9	T	E	10
2	3	4	5	6	7	8	9	T	E	10	11
3	4	5	6	7	8	9	T	E	10	11	12
4	5	6	7	8	9	T	E	10	11	12	13
5	6	7	8	9	T	E	10	11	12	13	14
6	7	8	9	T	E	10	11	12	13	14	15
7	8	9	T	E	10	11	12	13	14	15	16
8	9	T	E	10	11	12	13	14	15	16	17
9	T	E	10	11	12	13	14	15	16	17	18
T	E	10	11	12	13	14	15	16	17	18	19
E	10	11	12	13	14	15	16	17	18	19	1T

Check the following addition examples by using the addition table or by translating to decimal numerals. When you "carry" in adding duodecimal numerals, the sum must be greater than what number?

A. $\begin{array}{r} 6 \\ +\ T \\ \hline 14_{twelve} \end{array}$ B. $\begin{array}{r} T \\ +\ E \\ \hline 19_{twelve} \end{array}$ C. $\begin{array}{r} 36_{twelve} \\ +\ 75_{twelve} \\ \hline TE_{twelve} \end{array}$ D. $\begin{array}{r} 9T7_{twelve} \\ +\ E67_{twelve} \\ \hline 1952_{twelve} \end{array}$

We can now do base twelve subtractions; for example, if $7 + 8 = 13_{twelve}$, then $13_{twelve} - 7 = 8$, and $13_{twelve} - 8 = 7$. You can use your addition table to find $18_{twelve} - T$ by finding what number added to T gives 18_{twelve} as the sum: $18_{twelve} - T = ?$ You can also use a duodecimal nomograph to perform subtractions. As you study the following examples, check the subtraction by translating to base ten numerals.

A. $\begin{array}{r} 14_{twelve} \\ -\ 6 \\ \hline T_{twelve} \end{array}$ B. $\begin{array}{r} 3E_{twelve} \\ -\ 19_{twelve} \\ \hline 22_{twelve} \end{array}$ C. $\begin{array}{r} 52_{twelve} \\ -\ 28_{twelve} \\ \hline 26_{twelve} \end{array}$ D. $\begin{array}{r} 7T2_{twelve} \\ -\ 3E5_{twelve} \\ \hline 3T9_{twelve} \end{array}$ E. $\begin{array}{r} 604_{twelve} \\ -\ 2T8_{twelve} \\ \hline 318_{twelve} \end{array}$

EXERCISE SET 11

Base Twelve Addition and Subtraction

1. What are these sums?

a. $\begin{array}{r} 9 \\ +\ E \\ \hline 18 \end{array}$ b. $\begin{array}{r} T \\ +\ 8 \\ \hline 16 \end{array}$ c. $\begin{array}{r} 4E_{twelve} \\ +\ 57_{twelve} \\ \hline T6 \end{array}$ d. $\begin{array}{r} 64_{twelve} \\ +\ 58_{twelve} \\ \hline 100 \end{array}$

2. Add these numbers in base twelve:

a. $59_{twelve} + 58_{twelve}$ b. $28T_{twelve} + 139_{twelve}$ c. $7T63_{twelve} + 2095_{twelve}$ d. $5E4_{twelve} + 37T_{twelve} + E86_{twelve}$

3. Count by 7's in base twelve from T to $4E_{twelve}$.

4. Perform these subtractions in base twelve:

a. $15_{twelve} - 8$ b. $2T_{twelve} - 9$ c. $45_{twelve} - T$ d. $7T_{twelve} - E$

e. $7E_{twelve} - 35_{twelve}$ f. $25T_{twelve} - 176_{twelve}$ g. $520_{twelve} - 3E9_{twelve}$ h. $E05_{twelve} - TTT_{twelve}$

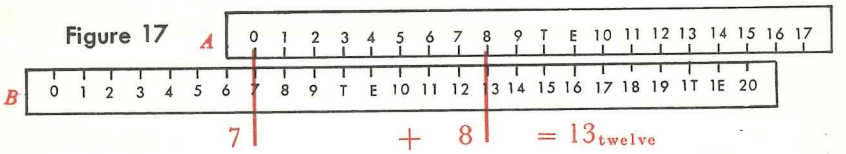
A Slide Rule for Duodecimal Addition and Subtraction

A duodecimal abacus with twelve beads on each wire, or the slide rule described below, will help you compute with duodecimals.

Cut two strips of cardboard about 2″ by 14″ each. On one edge of each strip locate 25 equally spaced points and mark them distinctly. Label the points with the numerals of the dozen system in order from 0 to 20_{twelve}. Mark edges that can be placed adjacent to each other.

To add 7 + 8, place the strips together and slide the *A* scale so that the 0 on *A* is even with the 7 on the *B* scale. Then the numeral on scale *B* under the 8 on the *A* scale is the sum, 7 + 8.

Figure 17



Thus, $7 + 8 = 13_{twelve}$. In a reverse manner, the slide can be used to find differences. Thus, to find $13_{twelve} - 8$, we move the 8 on the *A* scale directly over the 13 on the *B* scale. The answer, 7, is found on the *B* scale under the 0 on the *A* scale.

EXERCISE SET 12

Using the Duodecimal Slide Rule

Use your slide rule to perform the indicated operations:

1. 8 + T
2. $9 + 11_{twelve}$
3. $10_{twelve} + 10_{twelve}$
4. E − 6
5. $15_{twelve} - T$
6. $20_{twelve} - 12_{twelve}$

Duodecimal Multiplication and Division

Multiplication can be done by addition, with a duodecimal abacus, or by knowing the products of all duodecimal digits in the duodecimal notation. A simple way is to make a table of products and then refer to it whenever necessary. A good way to check your work is to convert to base ten numerals.

Copy and complete this multiplication table in base twelve.

Multiplication Table — Base Twelve

×	1	2	3	4	5	6	7	8	9	T	E
1	1	2	3	4	5	6	7	8	9	T	E
2	2	4	6	8	T						
3	3	6	9	10	13						
4	4	8	10	14	18						
5	5	T	13	18	21						
6	6	10	16	20	26	30					
7	7	12	19	24	2E		41				
8	8	14	20	28	34			54			
9	9	16	23	30					69		
T	T	18	26	34						84	
E	E	1T	29	38							T1

Study the following multiplication examples in base twelve.

A.	B.	C.	Check
23_{twelve}	507_{twelve}	48_{twelve} =	56
× 5	× $6T_{twelve}$	× 32_{twelve} =	×38
$E3_{twelve}$	425T	94	448
	2636	120	1680
	$2T5ET_{twelve}$	1294_{twelve} =	2128

In division with duodecimal numerals, we use addition, subtraction, and multiplication combinations. We also need to remember how multiplication and division are related. For example, if $7 \times 8 = 48_{twelve}$, then $48_{twelve} \div 7 = 8$, and $48_{twelve} \div 8 = 7$. Be sure to remember that when you carry or borrow you are using dozens, not tens. Try these divisions and compare your results with the examples below:

A. $54_{twelve} \div 8$ B. $T05_{twelve} \div 7$ C. $64T8_{twelve} \div 45_{twelve}$

Check your answers by translating to base ten numerals.

A. $8\overline{)54_{twelve}}$ — 8, 54

B. $7\overline{)T05_{twelve}}$ — $152_{twelve}\ r\ 3$; 7, 30, $2E$, 15, 12, 3

C. $45_{twelve}\overline{)64T8_{twelve}}$ — $154_{twelve}\ r\ 40_{twelve}$; 45, $1ET$, $1T1$, 198, 158, 40

Napier's Bones for Duodecimal Numerals

With the base twelve multiplication table, you can now make a set of Napier's bones for performing multiplications in this base. Cut thirteen strips of cardboard or plywood, each one inch by eleven inches. Mark twelve of the strips into twenty triangles and one square. Label each square with a different digit in the base twelve system. Then label the triangles with multiples of the digit in order. Your bone for 8_{twelve} will look like this:

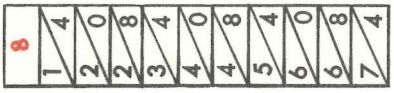


Figure 18

Make one strip an index strip, like this:

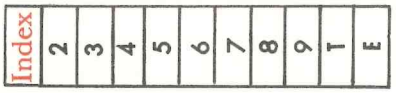


Figure 19

Index	8	E
2	1 4	1 T
3	2 0	2 9
4	2 8	3 8
5	3 4	5 7
6	4 0	5 6
7	4 8	6 5
8	5 4	7 4
9	6 0	8 3
T	6 8	9 2
E	7 4	T 1

Figure 20

Use the bones to multiply in problems such as 8E times T. Place the bones for 8 and E next to the index. Read the product in the T row of the index. Add the numbers along the diagonals to find the product.

EXERCISE SET 13

Base Twelve Multiplication and Division

1. Perform these multiplications with base twelve numerals. Check your answers with Napier's bones.

a. $\begin{array}{r} E \\ \times\ 7 \\ \hline \end{array}$

b. $\begin{array}{r} 23_{twelve} \\ \times\ 5 \\ \hline \end{array}$

c. $\begin{array}{r} 35_{twelve} \\ \times\ 6 \\ \hline \end{array}$

d. $\begin{array}{r} 87_{twelve} \\ \times\ 9 \\ \hline \end{array}$

e. $\begin{array}{r} 4T_{twelve} \\ \times\ 7 \\ \hline \end{array}$

f. $\begin{array}{r} 7E_{twelve} \\ \times\ 8 \\ \hline \end{array}$

g. $\begin{array}{r} 46_{twelve} \\ \times\ T \\ \hline \end{array}$

h. $\begin{array}{r} TE_{twelve} \\ \times\ 4 \\ \hline \end{array}$

2. Is T times E = E times T? Does the commutative law for multiplications hold for base twelve numbers?

3. Perform the indicated base twelve divisions. Check by multiplication or translation to base ten.

a. $5\overline{)164_{twelve}}$

b. $8\overline{)3648_{twelve}}$

c. $T\overline{)542_{twelve}}$

d. $23_{twelve}\overline{)57T6_{twelve}}$

Some Fascinating Fractions

Does the symbol $\frac{2}{3}$ have the same meaning in base twelve as in base ten? To answer this, we have to recall what the symbol $\frac{2}{3}$ means. You undoubtedly recognize this symbol as a *common fraction*, an expression of a part of a "whole thing." A drawing like the one in Figure 21 is often used to picture the meaning of $\frac{2}{3}$ It should now be evident that $\frac{2}{3}$ has the same meaning in base twelve as in base ten.

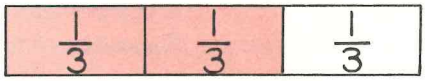


Figure 21

Let us now consider the base twelve fraction $\frac{5}{T}$. If we divide a rectangle into T equal parts and shade 5 of these parts, as in Figure 22, we see that $\frac{5}{T}$ represents one half of the rectangle. If

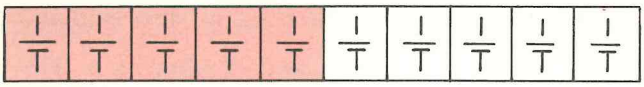


Figure 22

we reduce $\frac{5}{T}$ by division, $\frac{5 \div 5}{T \div 5}$, we find that $\frac{5}{T} = \frac{1}{2}$. This shows how base twelve and base ten fractions may look different but have the same meaning. Fractions in base twelve are different only because duodecimal numerals are used. Similarly,

$$\frac{7}{19_{\text{twelve}}} = \frac{7 \div 7}{19_{\text{twelve}} \div 7} = \frac{1}{3}.$$

By drawings, by reducing to lowest terms, or by knowing the value of duodecimal numerals, you can discover the meaning of a base twelve fraction.

By translating from one number base to another, we can compare fraction values.

Base ten		Base twelve
$\frac{5}{8}$	=	$\frac{5}{8}$
$\frac{7}{10}$	=	$\frac{7}{T}$
$\frac{11}{12}$	=	$\frac{E}{10_{\text{twelve}}}$
$\frac{13}{16}$	=	$\frac{11_{\text{twelve}}}{14_{\text{twelve}}}$

In base ten we find it very handy to use decimal fractions. We change common fractions in base ten numerals to decimals by division. For example, $\frac{3}{4} = 3 \div 4 = .75$. The decimal point is a short way of telling us that the denominator for .75 is one hundred.

In the same way, we change base twelve common fractions to *duodecimal* fractions. Duodecimal fractions are fractions with denominators such as twelve, or one gross. Instead of a decimal point, we use a *duodecimal point* (:). For example, :6 means six twelfths, or $\frac{6}{10_{\text{twelve}}}$; and $:26_{\text{twelve}}$ means $\frac{26_{\text{twelve}}}{100_{\text{twelve}}}$, or $\frac{30}{144}$ in base ten.

Let's express some common fractions as duodecimal fractions. The fraction $\frac{3}{4}$ can be interpreted as 3 divided by 4. The division in base twelve looks like this:

$$\begin{array}{r} :9 \quad \text{(Remember } 9 \times 4 = 30_{\text{twelve}}\text{)} \\ 4\overline{)3:0_{\text{twelve}}} \\ \underline{3\ 0} \end{array}$$

This tells us that $\frac{3}{4} = :9 = \frac{9}{10_{\text{twelve}}} = \frac{9}{12}$ (in base ten).

Let's use a different method to express $\frac{5}{9}$ as a base twelve rational. We first express $\frac{5}{9}$ as a base ten fraction whose denominator is a multiple of 12.

$$\begin{aligned} \frac{5}{9} &= \frac{80}{144} \quad \text{(in base ten)} \\ &= \frac{68_{\text{twelve}}}{100_{\text{twelve}}} \\ &= :68 \end{aligned}$$

One of the reasons base twelve is considered better than our base ten is that, as we have seen, twelve can be divided evenly by 2, 3, 4, and 6. Ten can be divided evenly only by 2 and 5. Thus, there are more base twelve common fractions that can be easily changed to duodecimal fractions than there are base ten common fractions that can be easily changed to decimal fractions.

Try changing other common fractions to duodecimal fractions in the exercises below and you will be surprised at the many simple answers.

EXERCISE SET 14

Working with Base Twelve Fractions

1. Change the following common fractions to duodecimal fractions:

a. $\frac{2}{3}$ b. $\frac{3}{2}$ c. $\frac{5}{6}$ d. $\frac{1}{4}$ e. $\frac{4}{12}$ f. $\frac{3}{8}$

2. Change the following from duodecimal to base ten fractions:

a. :5 b. :6 c. :8 d. :4 e. :24 f. 3:T

3. What denominators for base twelve fractions will give one-digit duodecimal equivalents?

Using Duodecimal Fractions

Duodecimals are very convenient in working with measures based on twelve, such as feet and inches. For example, 5 feet 7 inches = 5:7 feet, or 57_{twelve} inches. This simplifies computing with measured quantities like this:

Base ten addition	Base twelve addition
3 ft. 7 in.	$3{:}7_{twelve}$ ft.
+15 ft. 8 in.	+ $13{:}8_{twelve}$ ft.
18 ft. 15 in.	$17{:}3_{twelve}$ ft.
or 19 ft. 3 in.	

In base ten notation, comparisons involving division are often expressed in per cent. For example, 25% is interpreted as $\frac{25}{100}$.

In the duodecimal system, *per gross* is used instead of per cent. For example, 25_{twelve} per gross means $\frac{25_{twelve}}{100_{twelve}}$

If we translate to base ten numerals, $\frac{25_{twelve}}{100_{twelve}}$ becomes $\frac{29_{ten}}{144_{ten}}$

Computations are done the same way in both systems. When working with interest rates based on a twelve-month year, the duodecimal system is very convenient. But don't consider 6 per cent and 6 per gross as being equivalent rates! Which of these rates is the higher?

EXERCISE SET 15

Measurements in Base Twelve

1. Change the following measurements to duodecimal notation and compute as indicated. Check by using the usual computational method.

a. 7 ft. 8 in.
+ 8 ft. 9 in.

b. 5 ft. 3 in.
− 2 ft. 8 in.

c. 3 ft. 4 in.
× 2 ft. 7 in.

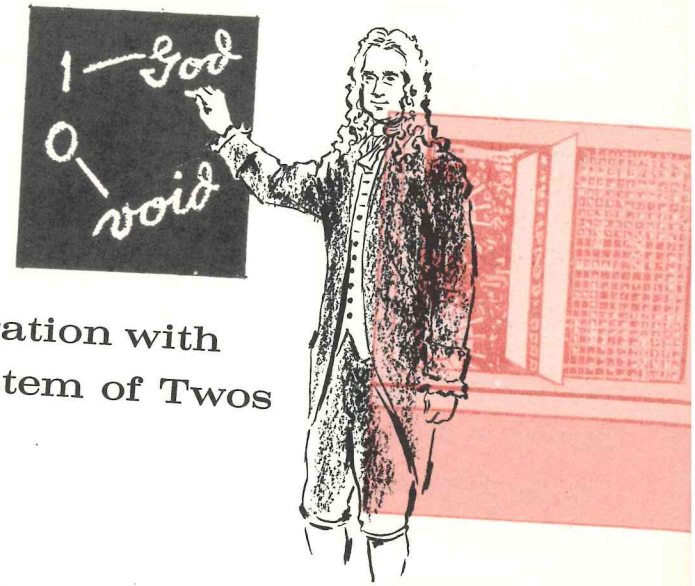
d. 9 ft. ÷ 2 ft. 3 in.

2. Express 37_{twelve} per gross as an indicated division of base twelve numbers and then change to base ten.

3. Which rate is higher:

a. 12 per cent or 12 per gross?

b. 5 per cent or 8 per gross?



Numeration with the System of Twos

Binary Numerals

During the latter part of the seventeenth century, a great German mathematician, Gottfried Leibniz, advocated the use of the simplest possible numeration system. This system needs only two symbols, 0 and 1, and is called a *binary system*. Leibniz was not the first person to work with a binary system, for certain types of binary systems had been used by even the most primitive peoples. But Leibniz, a very religious man, based his binary numeration system on a religious interpretation. He considered 1 to represent God, and 0 to represent the void, that apart from God. With the two symbols, 0 and 1, Leibniz could represent any number, just as he felt God could fashion the universe from the void.

Binary numerals were almost forgotten after Leibniz's death, but now they have been found to be very useful. The new computing machines, which we sometimes call *electronic brains*, use binary numerals to a great extent. The reason binary numerals are so useful in electronic brains is that 1 can be represented by an electric circuit with the switch "on" and 0 by the switch being "off."

Binary numerals are based on groups of twos, just as our everyday numeration system is based on tens. Let's see how base ten and base two numerals compare.

Base ten notation	Binary grouping	Binary notation	Suggested binary names
1	x	1	one
2	xx	10_{two}	twin
3	xx x	11_{two}	twin-one
4	xxxx	100_{two}	one twindred
5	xxxx x	101_{two}	one twindred one
6	xxxx xx	110_{two}	one twindred twin
7	xxxx xx x	111_{two}	one twindred twin one
8	xxxx xxxx	1000_{two}	one twosand
9	xxxx xxxx x	1001_{two}	one twosand one
10	xxxx xxxx xx	1010_{two}	one twosand twin
11	xxxx xxxx xx x	1011_{two}	one twosand twin one
12	xxxx xxxx xxxx	1100_{two}	one twosand one twindred
13	xxxx xxxx xxxx x	1101_{two}	one twosand one twindred one
14	xxxx xxxx xxxx xx	1110_{two}	one twosand one twindred twin
15	xxxx xxxx xxxx xx x	1111_{two}	one twosand one twindred twin one
16	xxxx xxxx xxxx xxxx	10000_{two}	twin twosand

We can use columns to keep place value straight for binary numerals and to change binary numerals to base ten notation. Let's work with 1101_{two}, 10110_{two}, 100000_{two}, and 111111_{two}.

two x two x two x two x two or twindred twosand or thirty-twos	two x two x two x two or twin twosand or sixteens	two x two x two or twosand or eights	two x two or twindred or fours	two x one or twins or twos	ones
		1	1	0	1
	1	0	1	1	0
1	0	0	0	0	0
1	1	1	1	1	1

$$1101_{\text{two}} = 1(\text{eight}) + 1(\text{four}) + 0(\text{two}) + 1 = 13$$

$$10110_{\text{two}} = 1(\text{sixteen}) + 0(\text{eight}) + 1(\text{four}) + 1(\text{two}) + 0 = 16 + 4 + 2 = 22$$

$$100000_{\text{two}} = 1(\text{thirty-two}) = 32$$

$$111111_{\text{two}} = 1(\text{thirty-two}) + 1(\text{sixteen}) + 1(\text{eight}) + 1(\text{four}) + 1(\text{two}) + 1 = 32 + 16 + 8 + 4 + 2 + 1 = 63$$

Electric Abacus for Base Two

A binary abacus like the one shown in Figure 23 with two beads or counters will help you keep place value straight and help you

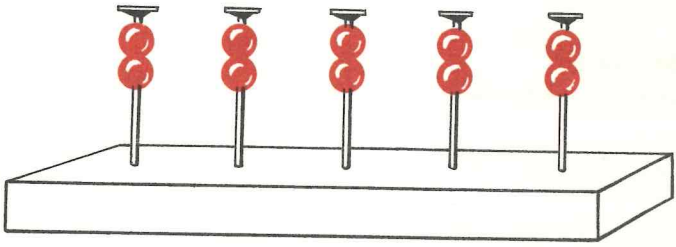


Figure 23

count with binary numerals. An electric abacus made according to the directions below can also be used.

Mount a string of Christmas tree lights evenly spaced on a board, as shown in Figure 24. Be sure the string is the kind where each light will burn independently of the others. When the string is plugged in, bulbs can be turned on or off by screwing the bulbs in or out. When a light is on, it stands for the digit 1. When the light is off, it represents 0.

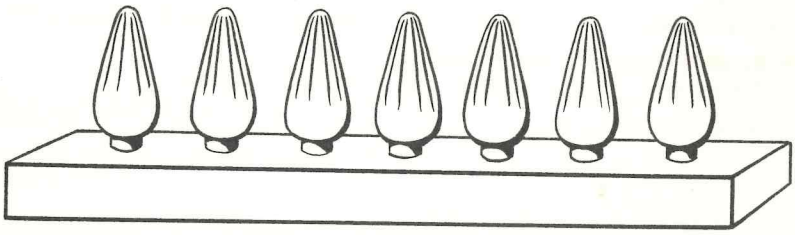


Figure 24

Figure 25 shows how the number 101_{two} looks with this abacus.

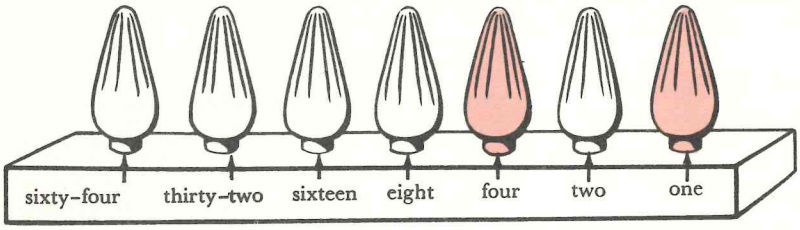


Figure 25

EXERCISE SET 16

Working with Binary Numerals

1. Copy and group these marks into twos, to form twins, twindreds, and twosands.

a. xxxxx b. xxxxx c. xxxxx d. xxxx
xxxxx xxxxx xxxxx
xxx

2. Write the binary number symbol for each of the numbers represented in Exercise 1 above.

3. Write the binary number names for each of the numbers represented in Exercise 1 above.

4. Make marks and group the marks to show place value for these binary numbers.

a. 110_{two} b. 1011_{two} c. 1100_{two} d. 10101_{two}

5. Write the binary notation for the numbers from 16 to 35.

6. What is the disadvantage in using binary notation?

From Base Ten to Binary Notation

We can change base ten numerals to binary numerals by using methods similar to the ones used to change from base ten to base twelve or base five. We can find the largest multiple of two in the number, subtract this multiple of two, and repeat the process with the remainder. Let's find the binary numeral for 13.

The highest multiple of two in thirteen is eight. Since eight is equivalent to two × two × two, thirteen can be written as a four-place binary numeral.

$$\begin{array}{r} 1 \\ 8\overline{)13} \\ \underline{8} \\ 5 \end{array}$$

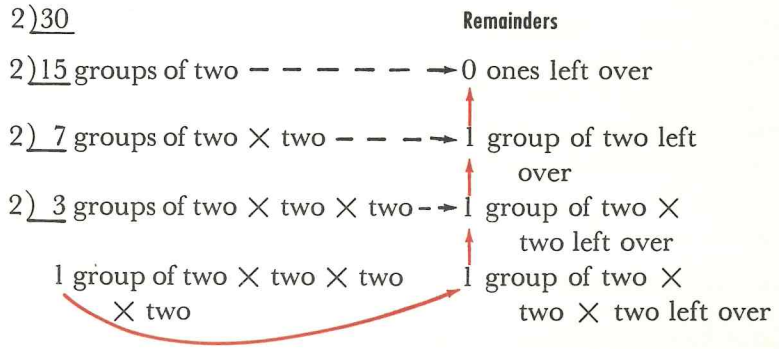
The division shows that there is one two × two × two group in thirteen. The subtraction shows that there are five units left over.

$$\begin{array}{r} 1 \\ 4\overline{)5} \\ \underline{4} \\ 1 \end{array}$$

The highest multiple of two in five is four, or two × two. There is one two × two group in five, and one unit left over. No groups of just two are needed to represent thirteen in binary notation.

We see that $13_{ten} = 1(two^3) + 1(two^2) + 0(two) + 1 = 1101_{two}$.

The example below shows how to use a short form of the above method to convert 30_{ten} to binary notation.


2)30
Remainders
2)15 groups of two
0 ones left over
2) 7 groups of two × two
1 group of two left over
2) 3 groups of two × two × two
1 group of two × two left over
1 group of two × two × two × two
1 group of two × two × two left over

Notice that we performed repeated divisions by 2; the remainders in reverse order gave the binary numeral 11110_{two} as equal to 30_{ten}.

EXERCISE SET 17

Binary Change of Base

1. Change these binary numerals to base ten numerals:
   a. 10101_{two} 21
   b. 110011_{two} 51
   c. 1011001_{two} 89
   d. 1111111_{two} 127
2. Change these base ten numerals to binary notation:
   a. 19 10011
   b. 37 100101
   c. 65 1000001
   d. 100 1100100
3. In what digit does an even number always end in binary notation? 0

Operations in Base Two

After writing the binary notation for numbers such as 30, namely 11110_{two}, you see that binary numerals are very long and tiresome to write. But computation with numbers expressed in the binary system is very simple. In addition, for example, all you need to know is that $1 + 0 = 1$ and $1 + 1 = 10_{two}$. No wonder some persons want us to adopt the binary numeration system! As you study the following examples, check the answers with base ten computation.

A.	B.	C.	D.	E.	Check
1	101_{two}	1	1011_{two}	$1111_{two} =$	15
$+\ 1$	$+\ 10_{two}$	1	$+\ 11_{two}$	$+\ 1011_{two} =$	$+\ 11$
10_{two}	111_{two}	$+\ 1$	1110_{two}	$11010_{two} =$	26
		11_{two}			

Since subtraction is the inverse of addition, we can easily subtract binary numerals. To do subtractions quickly, you need to memorize all the subtraction combinations. This is easily done because there are only three: $1 - 1 = 0$, $1 - 0 = 1$, and $10_{two} - 1 = 1$. Notice the simplicity of the following subtraction examples.

A.	B.	C.	D.	E.	Check
10_{two}	11_{two}	101_{two}	1101_{two}	$11001_{two} =$	25
$-\ 1$	$-\ 10_{two}$	$-\ 11_{two}$	$-\ 110_{two}$	$-\ 1011_{two} =$	11
1	1	10_{two}	111_{two}	$1110_{two} =$	14

Multiplication in binary numerals is also very simple. All you need to memorize is that $1 \times 0 = 0 \times 1 = 0$ and $1 \times 1 = 1$. You never get a two-digit numeral in binary multiplication of two one-digit numbers. This means you do not have to "carry" when multiplying numbers expressed in binary notation. Study these multiplication examples to see how base ten multiplication ideas are used in base two. Watch your zeros and your additions!

A.	B.	C.	Check
10_{two}	11_{two}	$1101_{two} =$	13
$\times\ 10_{two}$	$\times\ 11_{two}$	$\times\ 101_{two} =$	$\times\ 5$
100_{two}	11	1101	15
	11	1101	50
	1001_{two}	$1000001_{two} =$	65

Division with binary numerals is the same as division with decimal numerals. This is simple, too, because the quotient figure obtained is always 1 or 0. Study the examples to see how easy it is to do base two division.

A.

$$10_{two})\overline{11_{two}} \quad 1 \text{ r } 1$$
$$\underline{10}$$
$$1$$

B.

$$101_{two})\overline{11110_{two}} \quad 110_{two}$$
$$\underline{101}$$
$$101$$
$$\underline{101}$$

C.

$$110_{two})\overline{111111_{two}} \quad 1010_{two} \text{ r } 11_{two}$$
$$\underline{110}$$
$$111$$
$$\underline{110}$$
$$11$$

EXERCISE SET 18

Binary Operations

Perform the indicated operations using binary numerals. Change to decimal numerals to check.

1. a. $110_{two} + 11_{two}$ b. $1011_{two} + 11_{two}$ c. $1101_{two} + 111_{two}$ d. $10101_{two} + 1011_{two}$

2. a. $111_{two} - 10_{two}$ b. $101_{two} - 11_{two}$ c. $11101_{two} - 1011_{two}$ d. $1001_{two} - 111_{two}$

3. a. $11_{two} \times 10_{two}$ b. $101_{two} \times 11_{two}$ c. $111_{two} \times 101_{two}$ d. $1001_{two} \times 111_{two}$

4. a. $10_{two})\overline{10_{two}}$ c. $101_{two})\overline{11110_{two}}$

b. $10_{two})\overline{1010_{two}}$ d. $110_{two})\overline{111011_{two}}$

Base Two and Electronic Brains

We said earlier that the binary numeration system is often used by electronic computers, sometimes called electronic brains. These numerals are used because the two digits, 1 and 0, can be represented by a complete or closed circuit (1) or an open circuit (0). When an electric switch is closed, current flows and a light may go on. When a circuit is closed so that electricity flows, we use a 1 to denote this circuit. When a switch is open so that no current flows, we use a 0 to denote this circuit. The illustrations in Figure 26 show these circuits.

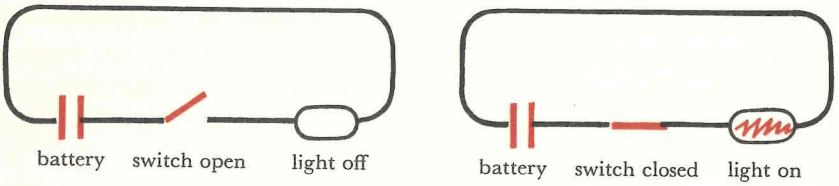
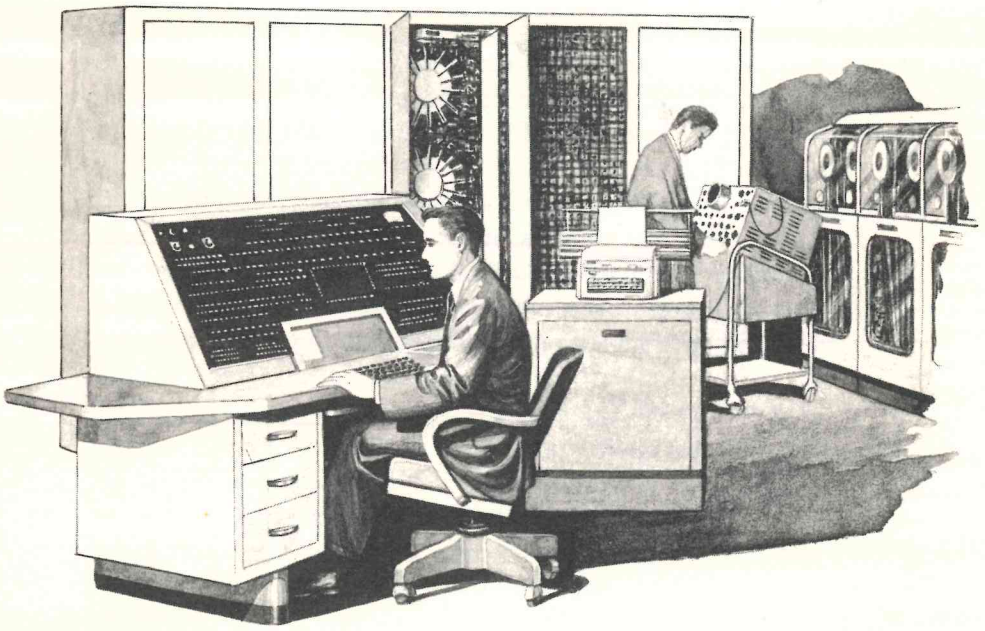


Figure 26

A number of these circuits arranged in order can represent binary numerals in the same manner as a set of Christmas tree lights in the binary abacus described on page 34.

Electronic brains can add, subtract, multiply, and divide numbers in binary notation as fast as the speed of light. The brains can be given instructions to solve complex mathematical and scientific problems. Computations for a jet engine that would normally require 3,000 hours can be done on an electronic computer in 2 hours. In solving these problems, binary numbers are used as a code to tell the computer what to do.

The actual operation of an electronic brain is very complex, but through a simple example we can get an idea of how the machine uses binary notation. Suppose we want the computer to subtract 13 from 27, and then add 9. Numerical values and instructions are usually put into the computer by magnetic tape or punched cards. We could punch holes in a card to represent



27, 13, and 9, and "feed" the values to the electronic brain. Inside the machine, seven circuits or switches (*a*, *b*, *c*, *d*, *e*, *f*, *g*) might be arranged in a definite order to represent binary numerals up to seven places. The following table shows how the circuits could represent 27, 13, and 9 in binary notation.

Base ten numerals	Binary numerals	Circuits						
		a	b	c	d	e	f	g
27	0011011			on	on		on	on
13	0001101				on	on		on
9	0001001				on			on

After "feeding" the numbers to the machine, we would have to tell the machine what to do with the numbers. Operational instructions, such as those shown in red below, would have to be given to the computer so that it could work the problem in a logical manner.

1. Subtract two numbers.
2. Write the difference.

$$27 - 13 = 14$$
$$14 + 9 = 23$$

3. Add to the difference.
4. Write the answer.

Each instruction must be represented with a numerical code. The instructions can then be punched on a card in the numerical code and fed into the computer in the proper order. Inside the electronic brain, the operational instructions would be changed to a binary code. The table below gives an idea of how the operations might be represented using another set of seven circuits in the computer. The computer is built so that numerals that represent operations can be distinguished from numerals to be operated on.

Operation	Binary numeral code	Circuits						
		a	b	c	d	e	f	g
Subtract two numbers.	0010101			on		on		on
Write the difference.	0001011				on		on	on
Add to the difference.	0110011		on	on			on	on
Write the answer.	1000000	on						

If everything was done properly, the binary numeral 0010111 would be obtained in a computer circuit. This could be translated to 23 and recorded on magnetic tape or punched in a card as the answer.

Electronic brains are no better than the mathematicians who use them. The mathematicians have to work out the problem to be solved in steps so that it can be put into the machine. After the method of solution is worked out, a programmer sets the problem up in code and in logical steps called a *program*. Figure 27 shows the program of a simple problem for one type of computer. In this way, electronic brains are solving all sorts of problems, from finding insurance rates to getting a space ship into orbit.

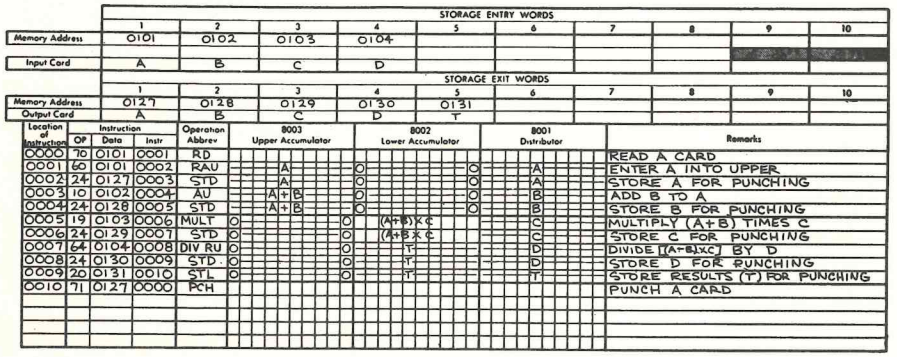

STORAGE ENTRY WORDS
Memory Address 0101 0102 0103 0104
Input Card A B C D
STORAGE EXIT WORDS
Memory Address 0127 0128 0129 0130 0131
Output Card A B C D T
Location of Instruction
Instruction OP Data Instr
Operation Abbrev
8003 Upper Accumulator
8002 Lower Accumulator
8001 Distributor
Remarks
0000 70 0101 0001 RD READ A CARD
0001 60 0101 0002 RAU ENTER A INTO UPPER
0002 24 0127 0003 STD STORE A FOR PUNCHING
0003 10 0102 0004 AU ADD B TO A
0004 24 0128 0005 STD STORE B FOR PUNCHING
0005 19 0103 0006 MULT MULTIPLY (A+B) TIMES C
0006 24 0129 0007 STD STORE C FOR PUNCHING
0007 64 0104 0008 DIV RU DIVIDE [(A+B)XC] BY D
0008 24 0130 0009 STD STORE D FOR PUNCHING
0009 20 0131 0010 STL STORE RESULTS (T) FOR PUNCHING
0010 71 0127 0000 PCH PUNCH A CARD

Figure 27. This is a computing machine program. It is set up to solve the equation $\frac{(A+B)\times C}{D} = T$. *Reprinted through the courtesy of International Business Machines Incorporated.*

Nim, a Binary Number Game

Base two numerals are very important in the internal operation of a computer, but they also have some other less practical, but just as interesting, applications. One such application is in a game called Nim.

This ancient mathematical game is played by two persons using sticks, matches, toothpicks, beads, coins, or other objects as counters. The counters are placed in piles and the players take turns picking up counters from the piles. The player who picks up the last counter wins. The number of piles used, the number of counters in each pile, and the number of counters picked up

at each turn may vary. The players may pick any number of counters at one time from one pile. They must not take counters from more than one pile at a given turn, and at least one counter must be taken on each turn. At the next turn, counters may be selected from a different pile, or, if it still remains, from the same pile. Remember, the player picking up the last counter is the winner. Consider this example:

Place 15 counters in three piles containing 3, 5, and 7 counters, as shown in Figure 28. Base two numerals can be used to plan the strategy.

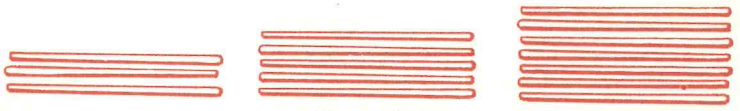


Figure 28

The game is analyzed by writing the number of counters in each pile in binary notation. The winner should always leave the piles so that the sum of the digits of every column of the binary numerals for each pile is an even number. If each sum of the column digits is always even after you pick up a counter, an even number of draws will have to be made to get all the counters, and you will win.

Number of counters in each pile	Binary notation			
	eights	fours	twos	ones
3			1	1
5		1	0	1
7		1	1	1
Column totals (base ten)		2	2	3

In order to win, the columns must be kept even. This may be done by picking up one counter from any pile which will make the column totals 2, 2, 2.

Consider another example. Place 24 counters in three piles, containing 9, 8, and 7 counters, respectively.

Number of counters in each pile		Binary notation
9	=	1001_{two}
8	=	1000_{two}
7	=	111_{two}
Column totals (base ten)		2112

You can put yourself in a winning position by taking six counters (110_{two}) from the third pile. Column totals are then all even, as shown below.

$$\begin{aligned} 9 &= 1001_{two} \\ 8 &= 1000_{two} \\ 1 &= \quad 1_{two} \\ \hline & \quad 2002 \end{aligned}$$

EXERCISE SET 19

Questions about Nim

1. If, at the beginning of a game of Nim, column totals are even when you express the counters in binary notation, would you want to draw first or second to insure victory?

2. If the column totals are odd at the start of the game, would you want to draw first or second to insure victory?

More Uses for Base Two

Another interesting application of the binary numeration system is a coin system that simplifies the task of making change. Instead of nickels, dimes, quarters, and half-dollars, a set of coins of value 1, 2, 4, 8, 16, 32, 64 cents is proposed. These new coins would give change for any amount up to $1.28 without using more than one coin of each denomination. For example, 83 cents = (64 + 16 + 2 + 1) cents.

In this new coin system, binary notation would be quite useful in making change. We could write an amount of change in binary notation, and then think of each place in the binary numeral as representing a coin with a value corresponding to the value of the place. A 1 in a place would tell us that a coin of that place value should be used in the change. For example, in binary notation, 83 is 1010011_{two}. This tells us that four coins, a 64, a 16, a 2, and a 1, are to be used to make change for 83 cents. With our present coins, we must often give several coins of the same denomination to make change; for example, 83 cents = 1 half-dollar, 1 quarter, 1 nickel, and 3 pennies. You would need to know binary numbers very well if we had this system of coins.

A clever trick based on binary numbers can be used to guess

your friend's age. For this trick, you need the cards shown in Figure 29.

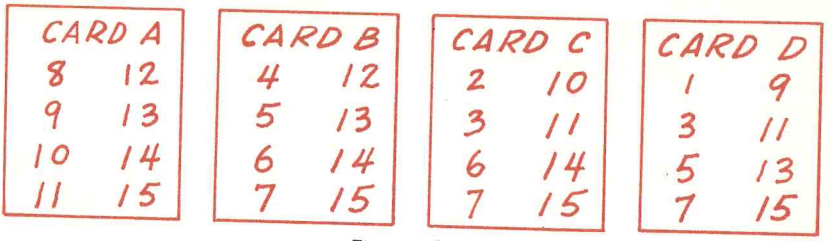


Figure 29

Ask your friend to name the cards on which his age is recorded. If he says A, B, and D, you tell him his age is 13. All you have to do to get his age is to add the numbers represented in the upper left-hand corner of the cards he reports. The numerals on the cards he selected are 8, 4, 1, and the sum is 13.

Binary notation is used to make the cards for this trick. Think of each card as representing the value of a place of a four-digit binary numeral. Card D represents 1, Card C represents two, Card B represents four, and Card A represents eight. Now write each base ten numeral from 1 to 15 in binary notation. A 1 in a digit place of the binary numeral indicates that the original base ten numeral is to be written on the card that corresponds to the value of that digit place. Thus, $8 = 1000_{\text{two}}$, $4 = 100_{\text{two}}$, $2 = 10_{\text{two}}$, and $1 = 1_{\text{two}}$, and 8, 4, 2, and 1 are written in the upper left-hand corner of cards A, B, C, and D respectively. Now 13 is written as 1101_{two}, and thus 13 must appear on Card A, Card B, and Card D. When your friend selects cards A, B, and D, you merely compute $8 + 4 + 1 = 13$.

EXERCISE SET 20

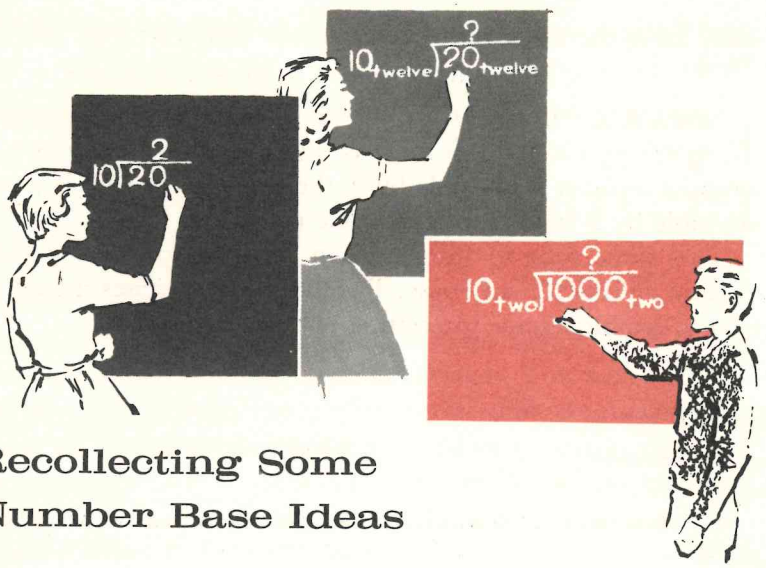
Using Base Two Numerals

1. With a set of coins of value 1, 2, 4, 8, 16, 32, and 64, what coins would be needed to make change for:

a. \$.46? b. \$1.02? c. \$1.12?

2. Make a set of cards that can be used to guess ages from 1 to 31.

3. The binary numbers can be used to make up a system of weights for a scale balance. What weights are needed to weigh any amount from 1 to 15 ounces? Be sure to find the fewest possible weights; no duplications are allowed.



Recollecting Some Number Base Ideas

Division Patterns

How can you tell which numbers written in base ten are divisible by 10, the base of the system? This question is an easy one, for you know that a number written in base ten is divisible by 10 if the units digit of its numeral is zero. Similarly, binary numbers whose numerals end in zero are known to be even numbers or numbers divisible by the base number two. Is it true that any number whose numeral ends in zero is divisible by its base number?

We can find other interesting tests for division in different number bases. Consider these four numbers written in base ten: 117, 414, 963, and 7884. Each number is divisible by 9. Now find the sum of the digits of each numeral and note your result. If you did your work correctly, you found that in each case the sum of the digits was 9 or a multiple of 9. We can show that this is not just a coincidence by examining the numeral 7884. This numeral could be written as

$$(7 \times 1000) + (8 \times 100) + (8 \times 10) + 4$$

or,

$$[7 \times (999 + 1)] + [8 \times (99 + 1)] + [8 \times (9 + 1)] + 4.$$

Applying the distributive principle, we have

$$(7 \times 999) + 7 + (8 \times 99) + 8 + (8 \times 9) + 8 + 4,$$

and, using the associative principle, we can rearrange this to give

$$(7 \times 999 + 8 \times 99 + 8 \times 9) + 7 + 8 + 8 + 4.$$

Now, 999, 99, and 9 are all divisible by 9, so the expression $(7 \times 999 + 8 \times 99 + 8 \times 9)$ is divisible by 9. Therefore, if the original number is divisible by 9, the sum $7 + 8 + 8 + 4$ must be divisible by 9 (and thus a multiple of 9), and it is.

This same type of thinking can be used to develop rules for certain divisions in any base. The following exercises should enable you to establish some interesting tests for divisibility.

EXERCISE SET 21

Tests for Divisibility

1. How can you tell which numbers written in base twelve are divisible by 6?

2. Write several numbers in base five which are divisible by 2. Is there an easy way to tell whether a number written in base five is divisible by 2?

3. How can you tell if a number written in base five is divisible by 4?

4. How can you tell if a number written in base twelve is divisible by 11?

A Backward Glance and a Look to the Future

In our exciting world of today scientists and scholars are discovering and inventing all kinds of new things. We have wonderful new medicines, new machines, new foods, and new homes. At the same time, mathematicians are devising new mathematics which will make possible many new inventions in the future. This booklet has described some of the number notations which may have many uses in the world of tomorrow.

The study of different numeration systems should have helped you to better understand our base ten system. You have seen the importance of a principle of grouping in different numeration systems, and can better appreciate the importance of grouping in the base ten system. You have seen how the place-value concept makes numerals of any numeration system easy to use and easy to write. Working with place value in other numeration systems should help you understand its meaning and importance in our base ten system.

You know how to add, subtract, multiply, and divide numbers written in base ten, but you have found that you could perform the same operations in base five, base twelve, and base two. Does $1 + 1$ always equal 2? Of course it does. But sometimes we have different ways of writing the answer. We may say $1 + 1 = 10_{\text{two}}$. Nevertheless, the basic principles involved in computations are the same in all numeration systems.

You know that $6 + 4 = 4 + 6$, but does $12_{\text{five}} + 14_{\text{five}} = 14_{\text{five}} + 12_{\text{five}}$? The affirmative answer to this question merely points out the fact that certain basic principles of mathematics apply to all numeration systems.

You have seen that our base ten system of numeration is not the only one possible. In fact, any number may be used as a base. Perhaps our system is not even the best. You may be the one to invent a different numeration method that will make many new discoveries possible. Other numeration systems already have many uses, but one of the most important uses is to help us better understand the numeration system we use every day — the base ten system.

EXERCISE SET 22

Making Your Own Numeration Systems

1. Make a new numeration system in which:

0 =

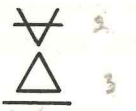
1 =

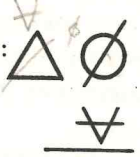
2 =

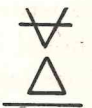
3 =

Write the numbers from 1 to 25 in this system. Make up a new set of number words.

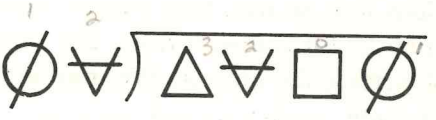
2. Try these problems:

a. Add: 

b. Subtract: 

c. Multiply: 

d. Divide:



3. Make up new number symbols with a number base other than that illustrated in this booklet.

4. Make up a new numeration system that does not use the principle of place value.

Numeration Review Test

A. Read each of the following statements carefully to determine whether it is a true or false statement:

1. The fraction "$\frac{2}{3}$" has the same value in the dozen system as in base ten.

2. The duodecimal ":5" has the same value as the decimal ".5".

3. We can make a number symbol mean anything that we want it to mean.

4. The numeral "8" (eight) has the same meaning in the duodecimal system as in our decimal system.

5. In performing computations there are more number combinations to remember in the base ten number system than in the base two.

10. When we "carry" a figure in an addition or multiplication problem, the value of the figure we "carry" is the same in base 12 as in base 10. F

11. The measurement 3 feet 7 inches can be written 3:7 feet in the duodecimal system.

12. When we borrow in the dozen system in a problem such as $154_{twelve} - 2T_{twelve}$, we actually borrow ten units. F

13. In the dozen system, 7 times 9 equals 53_{twelve}. T

14. In base five, $4 + 3 = 13_{five}$. F 12

15. The larger the number base the fewer the figures needed to represent large numbers. T

20. In the numeral 842_{twelve}, the 8 has a value that is twenty times the value of the 4. F

21. In base five, the fraction $\frac{3}{4}$ is equal to $\frac{14_{five}}{22_{five}}$.

B. Find the value of the expressions on the left in terms of the base ten numerals on the right. Match these answers in base ten with the numerals on the left. Any choice may be used once, several times, or not at all.

Expressions in base other than ten	Numbers in base ten
B 22. 10101_{two}	A. 31
B 23. 41_{five}	B. 21
A 24. 27_{twelve}	C. 16
25. $:6_{twelve}$	D. .6
C 30. $4_{five} \times 4_{five}$	E. .5
C 31. $1001_{two} + 111_{two}$	F. .4
32. $4_{six} \div 12_{six}$	
C 33. $32_{six} - 4_{six}$	
34. $\frac{1}{10_{two}}$	
35. $\frac{11_{five}}{20_{five}}$	

C. Perform the indicated operations:

40. Write 155_{ten} in base twelve. $10E_{(twelve)}$

41. Write 27_{ten} in base two. $11011_{(two)}$

42. Write 434_{ten} in base five. $3214_{(five)}$

43. Write 526_{ten} in base six. $2234_{(six)}$

44. Write 234_{twelve} in base ten. 328

45. Write 1324_{five} in base ten. 214

50. Write 101011_{two} in base ten. 43

D. Compute as indicated with these base five numerals. Leave the answers in base five.

51.
$$\begin{array}{r} 3202_{five} \\ +\ 4312_{five} \\ \hline \end{array}$$

52.
$$\begin{array}{r} 434_{five} \\ -\ 243_{five} \\ \hline \end{array}$$

53.
$$\begin{array}{r} 431_{five} \\ \times\ 32_{five} \\ \hline \end{array}$$

54. $4\overline{)121_{five}}$ = 14

E. Compute as indicated with these base twelve numerals:

55.
$$\begin{array}{r} 89T7_{twelve} \\ +\ E74_{twelve} \\ \hline \end{array}$$

100.
$$\begin{array}{r} 940E_{twelve} \\ -\ 34T_{twelve} \\ \hline \end{array}$$

101.
$$\begin{array}{r} 452_{twelve} \\ \times\ T8_{twelve} \\ \hline \end{array}$$

102. $8\overline{)57E_{twelve}}$ = 85 r 7

F. Compute as indicated in base two:

103.
$$\begin{array}{r} 1011_{two} \\ +\ 110_{two} \\ \hline \end{array}$$

104.
$$\begin{array}{r} 11011_{two} \\ -\ 1101_{two} \\ \hline \end{array}$$

105.
$$\begin{array}{r} 101_{two} \\ \times\ 110_{two} \\ \hline \end{array}$$

110. $10_{two}\overline{)11010_{two}}$ = 1101

G. Compute as indicated in base six:

111.
$$\begin{array}{r} 342_{six} \\ +\ 154_{six} \\ \hline \end{array}$$

112.
$$\begin{array}{r} 530_{six} \\ -\ 42_{six} \\ \hline \end{array}$$

113.
$$\begin{array}{r} 34_{six} \\ \times\ 5 \\ \hline \end{array}$$

114. $4\overline{)144_{six}}$

H. Perform the following:

115. Change the fraction $\frac{2}{3}$ to a duodecimal.

120. Change the duodecimal :3 to a base ten fraction.

I. Answer the following:

121. In which number base has this addition problem been worked?

$$\begin{array}{r} 5304 \\ +\ 2334 \\ \hline 12042 \end{array}$$

six

122. In which number base has this multiplication been performed?

$$\begin{array}{r} 123 \\ \times\ 32 \\ \hline 312 \\ 1101 \\ \hline 11322 \end{array}$$

four

123. What number base has been used in numbering the items of this test? six

Extending Your Knowledge

If you have enjoyed exploring the numeration systems in this booklet, you might want to consult some of these publications for further information:

ADLER, IRVING, *The Magic House of Numbers.* The John Day Co., 1957

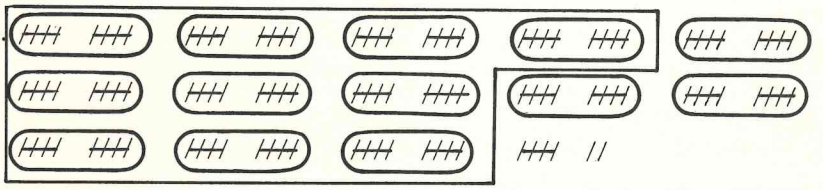
BAKST, AARON, *Mathematics, Its Magic and Mastery.* D. Van Nostrand Company, 1952

KASNER, EDWARD and NEWMAN, JAMES, *Mathematics and the Imagination.* Simon and Schuster, 1940

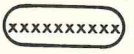
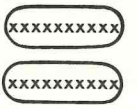
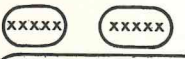
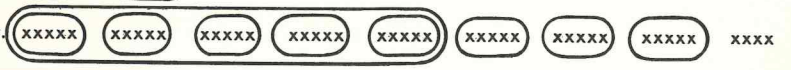
Meyer, Jerome, *Fun with Mathematics.* The World Publishing Co., 1952

Solutions to the Exercises

EXERCISE SET 1

1. *a.* 3000 *b.* $\frac{3}{10{,}000}$ *c.* 3/10
2. 432
3. 234
4. 
5. *a.* $(5 \times 10 \times 10 \times 10) + (6 \times 10 \times 10) + (8 \times 10) + (4 \times 1)$
   b. $(2 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (7 \times 10) + (0 \times 1)$
   c. $\frac{4}{10} + \frac{1}{10 \times 10} + \frac{3}{10 \times 10 \times 10}$
6. *a.* Because we have 10 fingers.
   b. Because they used 10 fingers and 10 toes for counting.

EXERCISE SET 2

1. *a.*  xxxx *b.*  xxx
   c. A group of ten groups of ten x's, three groups of ten x's, and seven x's
2. *a.* xxxx
   b.  xxx
   c. 
3. *a.* $(5 \times 10) + 3$
   b. $(1 \times 10 \times 10) + (2 \times 10) + 3$
   c. $(2 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (0 \times 10) + 4$
4. *a.* $(4 \times \text{five}) + 3$
   b. $(1 \times \text{five} \times \text{five}) + (2 \times \text{five}) + 3$
   c. $(2 \times \text{five} \times \text{five} \times \text{five}) + (3 \times \text{five} \times \text{five}) + (0 \times \text{five}) + 4$
5. *a.* $(2 \times \text{five}) + 4 = 14$
   b. $(4 \times \text{five} \times \text{five}) + (4 \times \text{five}) + 1 = 121$
   c. $(1 \times \text{five} \times \text{five} \times \text{five}) + (2 \times \text{five} \times \text{five}) + (0 \times \text{five}) + 3 = 178$

EXERCISE SET 3

1. *a.* 123_{five} *b.* 224_{five} *c.* 1111_{five} *d.* 3002_{five}
2. *a.* 13 *b.* 62 *c.* 95 *d.* 253

EXERCISE SET 4

1. 2 3 4 10 11
   3 4 10 11 12
   4 10 11 12 13
   10 11 12 13 14
   11 12 13 14 20
2. *a.* 44_{five} *b.* 41_{five} *c.* 112_{five} *d.* 1003_{five}
3. A ten 4. A five

EXERCISE SET 5

1. *a.* 21_{five} *b.* 13_{five} *c.* 223_{five} *d.* 3124_{five}
2. Ten 3. Five

EXERCISE SET 6

1. *a.* 20_{five} *b.* 30_{five} *c.* 31_{five}
2. *a.* 3_{five} *b.* 3_{five} *c.* 4_{five}

EXERCISE SET 7

1. *a.* 201_{five} *b.* 332_{five} *c.* 1442_{five} *d.* 4431_{five}
2. *a.* Yes *b.* Yes *c.* Yes
3. *a.* Yes *b.* Yes *c.* Yes
4. Yes

EXERCISE SET 8

1. 112_{five} 2. 40_{five} 3. 21_{five} 4. 104_{five}

EXERCISE SET 9

1. 53 2. 9 3. 43 4. 28

EXERCISE SET 10

1. *a.* 66_{twelve} *b.* $9T_{twelve}$ *c.* 311_{twelve} *d.* 2234_{twelve}
2. *a.* 92 *b.* 131 *c.* 377 *d.* 264

EXERCISE SET 11

1. *a.* 18_{twelve} *b.* 16_{twelve} *c.* $T6_{twelve}$ *d.* 100_{twelve}
2. *a.* $E5_{twelve}$ *b.* 407_{twelve} *c.* $9E38_{twelve}$ *d.* 1938_{twelve}
3. 15, 20, 27, 32, 39, 44, 4E
4. *a.* 9 *b.* 21_{twelve} *c.* 37_{twelve} *d.* $6E_{twelve}$
   e. 46_{twelve} *f.* $T4_{twelve}$ *g.* 123_{twelve} *h.* 17_{twelve}

EXERCISE SET 12

1. 16_{twelve} 2. $1T_{twelve}$ 3. 20_{twelve}
4. 5 5. 7 6. T

EXERCISE SET 13

1. *a.* 65_{twelve} *b.* $E3_{twelve}$ *c.* 186_{twelve} *d.* 653_{twelve}
   e. $29T_{twelve}$ *f.* 534_{twelve} *g.* 390_{twelve} *h.* 378_{twelve}
2. Yes
3. *a.* 38_{twelve} *b.* 537_{twelve} *c.* 65_{twelve} *d.* 262_{twelve}

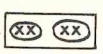
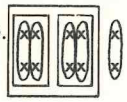
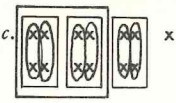
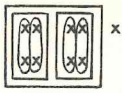
EXERCISE SET 14

1. *a.* :8 *b.* 1:6 *c.* :T *d.* :3 *e.* :4 *f.* :46
2. *a.* 5/12 *b.* 6/12 or 1/2 *c.* 8/12 or 2/3 *d.* 1/3 *e.* 28/144 or 7/36 *f.* 3 10/12 or 3⅚
3. 2, 3, 4, 6, 10_{twelve}

EXERCISE SET 15

1. *a.* 145_{twelve} inches *b.* 27_{twelve} inches *c.* 874_{twelve} inches *d.* 4
2. $\frac{37_{twelve}}{100_{twelve}} = \frac{43}{144} = 29.9\%$
3. *a.* 12% *b.* 8 per gross

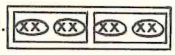
EXERCISE SET 16

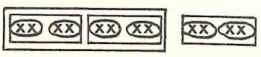
1. *a.*  *b.*  *c.*  *d.* 

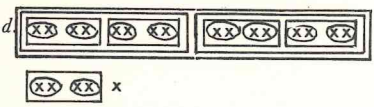
2. *a.* 101 *b.* 1010 *c.* 1101 *d.* 1001

3. *a.* One twindred one
   b. One twosand twin
   c. One twosand one twindred one
   d. One twosand one

4. *a.*

   b. 

   c. 

   d. 

5. 10000, 10001, 10010, 10011, 10100, 10101, 10110, 10111, 11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111, 100000, 100001, 100010, 100011

6. Small numbers use many digits.

EXERCISE SET 17

1. *a.* 21	*b.* 51	*c.* 89	*d.* 127
2. *a.* 10011_{two}	*b.* 100101_{two}	*c.* 1000001_{two}	*d.* 1100100_{two}

3. 0

EXERCISE SET 18

1. *a.* 1001_{two}	*b.* 1110_{two}	*c.* 10100_{two}	*d.* 100000_{two}
2. *a.* 101_{two}	*b.* 10_{two}	*c.* 10010_{two}	*d.* 10_{two}
3. *a.* 110_{two}	*b.* 1111_{two}	*c.* 100011_{two}	*d.* 111111_{two}
4. *a.* 1	*b.* 101_{two}	*c.* 110_{two}	*d.* $1001r101_{\text{two}}$

EXERCISE SET 19

1. Second
2. First

EXERCISE SET 20

1. *a.* 32, 8, 4, 2 *b.* 64, 32, 4, 2 *c.* 64, 32, 16
2. Cards like those in Figure 29: 5 cards, 16 numbers on each.
3. 1, 2, 4, 8

EXERCISE SET 21

1. If the units digit is 6 or 0, the number is divisible by 6.
2. Yes. If the sum of the last two digits is an even number, the number is divisible by 2.
3. If the sum of the digits is divisible by 4.
4. If the sum of the digits is divisible by 11(E).

EXERCISE SET 22

1. ϕ, $\forall$, $\triangle$, $\phi\square$, $\phi\phi$, $\phi\forall$, $\phi\triangle$, $\forall\square$, $\forall\phi$, $\forall\forall$, $\forall\triangle$, $\triangle\square$, $\triangle\phi$, $\triangle\forall$, $\triangle\triangle$, $\phi\square\square$, $\phi\square\phi$, $\phi\square\forall$, $\phi\square\triangle$, $\phi\phi\square$, $\phi\phi\phi$, $\phi\phi\forall$, $\phi\phi\triangle$, $\phi\forall\square$, $\phi\forall\phi$.
2. *a.* $\phi\phi$ *b.* $\forall\triangle$ *c.* $\phi\forall$ *d.* $\forall\phi\phi r\triangle$

3 and 4. Check your invention with a mathematics teacher.

NUMERATION REVIEW TEST

1. T	2. F	3. T	4. T
5. T	10. F	11. T	12. F
13. T	14. F	15. T	20. F
21. T	22. B	23. B	24. A
25. E	30. C	31. C	32. E
33. C	34. E	35. D	40. $10E_{twelve}$
41. 11011_{two}	42. 3214_{five}	43. 2234_{six}	44. 328
45. 214	50. 43	51. 13014_{five}	52. 141_{five}
53. 30342_{five}	54. 14_{five}	55. $995E_{twelve}$	100. 9081_{twelve}
101. $3E314_{twelve}$	102. 85_{twelve}, 7	103. 10001_{two}	104. 1110_{two}
105. 11110_{two}	110. 1101_{two}	111. 540_{six}	112. 444_{six}
113. 302_{six}	114. 24_{six}	115. :8	120. $\frac{1}{4}$
121. Six	122. Four	123. Six	

5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 — VH — 70 69 68 67 66 65 64 63 62

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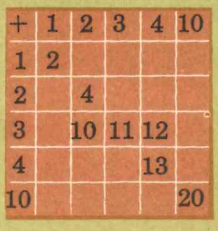

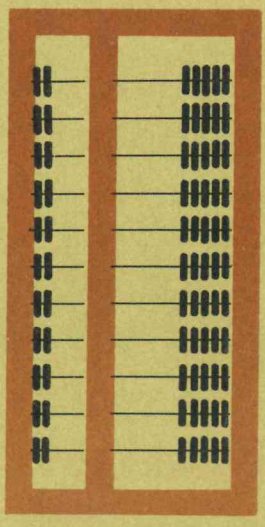

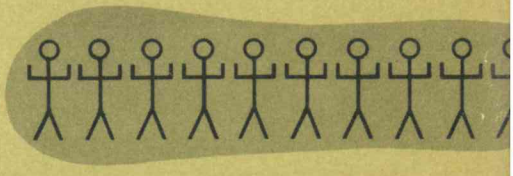

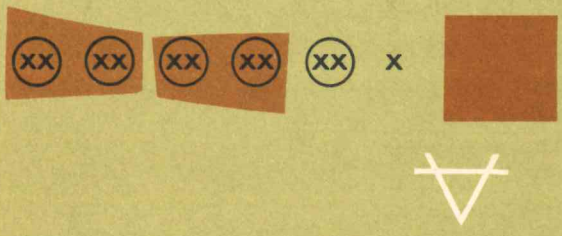

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